Spanning Trees, Modular Symbols, and a New Arithmetic Invariant of Elliptic Curves

Research Notes

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Abstract

Starting from the spanning tree spectrum of Alon–Bucić–Gishboliner, we construct a dictionary between feasible vectors of series-parallel graphs and cusps of $\Gamma_0(N)$. For any weight-2 newform f of level N, the average of the plus modular symbol $\{0, t/u\}^+$ over all feasible vectors of weight n converges to a limit $c_f \in \mathbb{Q}$ as $n \to \infty$. We prove this by reducing the problem to a finite Markov chain on the cusp graph $(\mathbb{Z}/N\mathbb{Z})^2$, and show that c_f is given by an explicit rational linear algebra formula.

The product $\lambda_f = c_f \cdot \Omega^+$ is an invariant of the isogeny class, defining a new arithmetic invariant of weight-2 newforms. We compute λ_f for all 93 elliptic curves (one per isogeny class) of conductor ≤ 100 and establish that the denominators of c_f are governed by $\det(I - P + \Pi)$, a purely combinatorial quantity depending only on N. The "alien primes" appearing in $\mathrm{den}(c_f)$—primes dividing neither N, $|E_{\mathrm{tors}}|$, nor 6—are identified as spectral invariants of the Markov chain acting on $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

For the rank-0 case at prime level, we prove that the key second moment $s_2 = \sum k^2 \psi(k) > 0$ via a novel polynomial factorization: the generating function $H(x) = \sum \psi(r) x^r$ factors as $x(1-x)^2 P(x)$ where P has all non-negative coefficients, verified for all rank-0 optimal curves of prime conductor $p \leq 200$.

Contents

1 Introduction

Let $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. Following Alon, Bucić, and Gishboliner [1], a vector $\left(\begin{smallmatrix}t\\u\end{smallmatrix}\right) \in \mathbb{Z}^2$ is *feasible* of weight n if

$$\begin{pmatrix}t\\u\end{pmatrix} = A^{a_1}D^{b_1}\cdots A^{a_\ell}D^{b_\ell}\begin{pmatrix}1\\0\end{pmatrix} \tag{1}$$

for some $a_i, b_i \geq 1$ with $\sum(a_i+b_i) = n$. The set $\mathcal{F}_n$ of feasible vectors satisfies $|\mathcal{F}_n| = 2^{n-1}-1$, and $t/u = [a_1; b_1, a_2, \ldots, b_\ell]$ as a continued fraction.

Each feasible vector corresponds to a two-terminal series-parallel graph G_w where $w = A^{a_1}D^{b_1}\cdots A^{a_\ell}D^{b_\ell}$: the number t counts spanning trees and t/u is the effective resistance with unit weights.

Given a weight-2 newform f of level N, denote by $\{0,r\}^+$ the plus-part modular symbol, normalized so that $\{0,r\}^+ = \frac{1}{\Omega^+}\operatorname{Re}\int_0^r 2\pi i f(z)\,dz$. Our objects of study are the normalized averages

$$c_f(n) \;=\; \frac{1}{|\mathcal{F}_n|} \sum_{(t,u)\in\mathcal{F}_n} \{0, t/u\}^+.$$

2 Main Results

Theorem 2.1 (Rationality)**.** *For every weight-2 newform f of level N, the limit $c_f = \lim_{n\to\infty} c_f(n)$ exists and is a rational number. Explicitly,*

$$c_f \;=\; \mu^T (I - P + \Pi)^{-1} \psi\,, \tag{2}$$

where $P \in \mathbb{Q}^{S\times S}$ is the transition matrix of the cusp Markov chain, $\psi \in \mathbb{Q}^S$ is the Manin symbol increment vector, $\mu \in \mathbb{Q}^S$ is the initial distribution, $\Pi = \mathbf{1}\pi^T$ is the projection onto the stationary distribution, and $S = \{(c, d) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(c, d, N) = 1,\ (c, d) \neq (0, 0)\}$.

Theorem 2.2 (Isogeny invariance)**.** *Let E, E' be elliptic curves in the same isogeny class over $\mathbb{Q}$, with associated newform f. Then $c_f(E)\cdot\Omega^+(E) = c_f(E')\cdot\Omega^+(E')$. The quantity $\lambda_f := c_f \cdot \Omega^+$ depends only on the isogeny class.*

Theorem 2.3 (Denominator theorem)**.** *For every weight-2 newform f of level N,*

$$\mathrm{den}(c_f) \;\Big|\; \det(I - P + \Pi),$$

where P and Π are as in Theorem 2.1. The matrix $I - P + \Pi$ depends only on N (not on f), so the primes that can appear in $\mathrm{den}(c_f)$ are determined by N alone.

Theorem 2.4 (Trivial graph motive)**.** *For any series-parallel graph G_w, the Kirchhoff polynomial Ψ_{G_w} is multilinear in the parallel-edge variables and independent of the series-edge variables. The graph hypersurface X_{G_w} has trivial motive, the Brown–Schnetz c_2 invariant vanishes, and the Feynman period is a product of beta functions.*

Theorem 2.5 (Non-universality)**.** *The ratio $\lambda_f/L(f, 1)$ is rational for rank-0 curves but takes distinct values across different newforms: it is not a universal function of $L(f, 1)/\Omega^+$, $|E_{\mathrm{tors}}|$, Tamagawa numbers, or the conductor N.*

Theorem 2.6 (Cusp evaluation)**.** *Let $V = (I - P + \Pi)^{-1}\psi_f$ be the value function. Then:*

(i) $V([1 : 0]) = c_f$,

(ii) $V([0 : 1]) = c_f - L(f, 1)/\Omega_f^+$.

In particular, c_f is the expected total Manin symbol accumulated by the chain started at the cusp $[1 : 0] = \infty$, and positivity $c_f > 0$ is equivalent to $V(\infty) > 0$.

Theorem 2.7 (Positivity)**.** *$c_f > 0$ for all 93 optimal newforms of conductor $N \leq 100$.*

Theorem 2.8 (Steinberg irreducibility)**.** *For prime $N = p$, the representation of the transition operator P on $\mathbb{P}^1(\mathbb{F}_p)$ decomposes as $\mathbf{1} \oplus \mathrm{St}_p$, where St_p is the Steinberg representation of dimension p. The characteristic polynomial of $(2^p - 1) \cdot P|_{\mathrm{St}_p}$ is irreducible of degree p over $\mathbb{Q}$, with Galois group S_p.*

Theorem 2.9 (Mersenne denominator)**.** *For prime $N = p$,*

$$\det(I - P|_{\mathrm{St}_p}) \;=\; \frac{A_p}{\Phi_p(2)}, \qquad \Phi_p(2) = 2^p - 1,$$

where $A_p \in \mathbb{Z}$ and Φ_p denotes the p-th cyclotomic polynomial. The alien primes at level p are exactly the odd prime factors of A_p exceeding 3.

Theorem 2.10 (Alien primes as norms)**.** *Let α be any root of the Steinberg characteristic polynomial at prime level p, and let $K_p = \mathbb{Q}(\alpha)$. Then*

$$A_p = \mathrm{N}_{K_p/\mathbb{Q}}\big((2^p - 1)(1 - \alpha)\big),$$

and the alien primes are exactly the primes dividing this norm that do not divide $6p(2^p - 1)$.

Theorem 2.11 (Second moment positivity)**.** *For every rank-0 optimal elliptic curve $E/\mathbb{Q}$ of prime conductor $p \le 200$, the second moment $s_2 = \sum_{k=1}^{p-1} k^2 \psi([1:k])$ is strictly positive. The positivity is established via a polynomial factorization: the generating function $H(x) = \sum_{r=1}^{p-1} \psi(r)\, x^r$ satisfies $H(x) = x(1-x)^2 P(x)$ where $P(x)$ has all non-negative coefficients.*

3 Proof of Rationality

The proof proceeds in five steps.

3.1 Translation invariance

Lemma 3.1. *$\{0, a + p/q\}^+ = \{0, p/q\}^+$ for all integers $a \ge 0$ and rationals p/q.*

Proof. Since $z \mapsto z + a$ lies in $\mathrm{SL}_2(\mathbb{Z})$ and maps $\{a, a+p/q\}$ to $\{0, p/q\}$, we have $\{a, a+p/q\}^+ = \{0, p/q\}^+$. Also $\{0, a\}^+ = 0$ since a is $\Gamma_0(N)$-equivalent to $i\infty$. □

Since $a_1 = \lfloor t/u \rfloor \ge 1$ for any feasible vector, $\{0, t/u\}^+ = \{0, \{t/u\}\}^+$.

3.2 The increment function

The CF expansion of $\{t/u\} = [0; b_1, a_2, \ldots, b_\ell]$ has convergents p_k/q_k. By telescoping,

$$\{0, \{t/u\}\}^+ = \sum_{k=1}^{L} \psi_k, \qquad \psi_k = \{0, p_k/q_k\}^+ - \{0, p_{k-1}/q_{k-1}\}^+.$$

Lemma 3.2. *The increment ψ_k depends only on $(q_{k-1} \bmod N,\ q_k \bmod N)$.*

Proof. The convergent matrix $\gamma_k = \left(\begin{smallmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{smallmatrix}\right)$ has $\det \gamma_k = (-1)^k$. In the Manin symbol formalism, the path $\{p_{k-1}/q_{k-1},\ p_k/q_k\}$ corresponds to the symbol $[\gamma_k] \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Via the isomorphism $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ given by $\gamma \mapsto$ (bottom row of γ) mod N, the symbol $[\gamma_k]$ is determined by $(q_{k-1} \bmod N,\ q_k \bmod N)$. Since $-I$ acts trivially on weight-2 forms, the sign ambiguity $\det \gamma_k = \pm 1$ does not affect ψ_k^+. □

Define $\psi \colon S \to \mathbb{Q}$ by this consistent value.

3.3 The Markov chain

The CF recurrence $q_{k+1} = a_{k+1} q_k + q_{k-1}$ gives the transition $(q_{k-1} \bmod N,\ q_k \bmod N) \to (q_k \bmod N,\ (a_{k+1} q_k + q_{k-1}) \bmod N)$.

The state space is $S = \{(c, d) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(c, d, N) = 1\} \setminus \{(0,0)\}$. For large weight, the partial quotients are asymptotically iid Geom(1/2). Grouping $j \ge 1$ by residue $r = j \bmod N$:

$$P\big((c,d) \to (d,e)\big) = \sum_{\substack{j \ge 1 \\ jd + c \equiv e \pmod N}} 2^{-j} = \frac{2^{N-r}}{2^N - 1}, \tag{3}$$

where $r \in \{0, 1, \ldots, N-1\}$ satisfies $rd + c \equiv e \pmod N$.

3.4 Double stochasticity and centering

Lemma 3.3. *P is doubly stochastic on S.*

Proof. For target state (d, e), each column entry $P((c, d) \to (d, e))$ corresponds to a unique residue $r = r(c) \in \{0, \dots, N-1\}$ (namely r such that $rd + c \equiv e \pmod{N}$), and conversely each r determines a unique c. So the column sum is $\sum_{r=0}^{N-1} \frac{2^{N-r}}{2^N-1} = \frac{2^N+2^{N-1}+\dots+2^1}{2^N-1} = 1$. □

Lemma 3.4. $\mathbb{E}_\pi[\psi] = \sum_{s\in S} \pi(s)\psi(s) = 0$.

Proof. Since P is doubly stochastic, π is uniform: $\pi(s) = 1/|S|$. So $\mathbb{E}_\pi[\psi] = 0$ iff $\sum_{s\in S} \psi(s) = 0$. The group $(\mathbb{Z}/N\mathbb{Z})^\times$ acts freely on S by $\lambda \cdot (c, d) = (\lambda c, \lambda d)$. Each orbit maps to a single projective point $[c : d] \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, and each fiber has size exactly $\varphi(N)$.

Since ψ is constant on $(\mathbb{Z}/N\mathbb{Z})^\times$-orbits (the Manin symbol depends only on the projective class), $\sum_{s\in S} \psi(s) = \varphi(N) \sum_{[c:d]\in\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} [c : d]_f^+$. The classical Manin relation gives $\sum_{[c:d]} [c : d]_f^+ = 0$, completing the proof. □

3.5 Spectral gap

Lemma 3.5. *P restricted to S is irreducible and aperiodic.*

Proof. Irreducibility. With $j = 0$: $T_0(c, d) = (d, c)$; with $j = 1$: $T_1(c, d) = (d, d + c)$. These correspond to right-multiplication by $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $SU = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, which generate $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ acts transitively on S, the chain is irreducible.

Aperiodicity. For $c \not\equiv 0 \pmod{N}$, the state (c, c) has a self-loop with $j = 0$: $(c, c) \to (c, 0 \cdot c + c) = (c, c)$, with weight $w_0 = 1/(2^N - 1) > 0$. □

Since $\mathbb{E}_\pi[\psi] = 0$ and P has a spectral gap, the Neumann series $V = \sum_{k\geq 0} P^k \psi$ converges. Equivalently, $V = (I - P + \Pi)^{-1}\psi$, which is rational since P, Π, and ψ are all rational.

3.6 The formula for c_f

The initial state is $X_0 = (1, b_1 \bmod N)$ with $b_1 \sim \mathrm{Geom}(1/2)$. The initial distribution $\mu(1, r) = 2^{N-r}/(2^N - 1)$ is rational. Then

$$c_f = \sum_s \mu(s)V(s) = \mu^T (I - P + \Pi)^{-1}\psi \;\in\; \mathbb{Q}.$$

4 Proof of the Denominator Theorem

Proof of Theorem 2.3. By Cramer's rule, $(I - P + \Pi)^{-1} = \mathrm{adj}(I - P + \Pi)/\det(I - P + \Pi)$. The adjugate matrix has rational entries. Since μ and ψ are also rational,

$$c_f = \frac{\mu^T \cdot \mathrm{adj}(I - P + \Pi) \cdot \psi}{\det(I - P + \Pi)} \;\in\; \mathbb{Q},$$

and $\mathrm{den}(c_f) \mid \det(I - P + \Pi)$ (up to cancellation in the numerator).

Crucially, P and Π depend only on N: the transition matrix (3) uses only the weights $2^{N-r}/(2^N-1)$ and the group law of $(\mathbb{Z}/N\mathbb{Z})^2$, while $\Pi = \mathbf{1}\pi^T$ with π uniform on S. The Manin symbol vector ψ varies with f, but it enters only in the numerator. □

Proof of Theorem 2.6. The value function $V = (I - P + \Pi)^{-1}\psi$ satisfies $(I - P + \Pi)V = \psi$, so for each state s,

$$V(s) = \psi(s) + \sum_t P(s, t)V(t) - \frac{1}{|\mathbb{P}^1|} \sum_t V(t). \tag{4}$$

We first show the last term vanishes. Since P is stochastic, $\mathbf{1}^T P = \mathbf{1}^T$ and $\mathbf{1}^T \Pi = \mathbf{1}^T$, so $\mathbf{1}^T(I-P+\Pi) = \mathbf{1}^T$. Multiplying $(I-P+\Pi)V = \psi$ on the left by $\mathbf{1}^T$ gives $\sum_t V(t) = \sum_t \psi(t) = 0$ (centering). So (4) simplifies to

$$V(s) = \psi(s) + [PV](s). \tag{5}$$

From $[1:0]$: the transition sends $(c,d) = (1,0) \mapsto (d, rd+c) = (0,1)$ for every partial quotient r, so $P([1:0],\cdot) = \delta_{[0:1]}$. Thus $V([1:0]) = \psi([1:0]) + V([0:1])$.

From $[0:1]$: the transition sends $(0,1) \mapsto (1,r)$ with weight w_r for each r, so $P([0:1],[1:r]) = w_r$. Thus $V([0:1]) = \psi([0:1]) + \sum_{r=0}^{N-1} w_r V([1:r]) = \psi([0:1]) + c_f$, where the last equality holds because $c_f = \mu^T V = \sum_r w_r V([1:r])$.

Adding and using the S-relation $\psi([1:0]) + \psi([0:1]) = 0$: $V([1:0]) = c_f$. □

Remark 4.1. The identity $V([0:1]) = c_f - L(f,1)/\Omega^+$ follows from $\psi([0:1]) = -L(f,1)/\Omega^+$ (since $[0:1]$ is the cusp 0 and $\psi([0:1]) = \{0,\infty\}^+$ is the period integral).

Proof of Theorem 2.7. The constraint $\sum_{s\in S} V(s) = 0$ and the cusp evaluation give

$$c_f = \frac{L(f,1)/\Omega^+ - G}{n}, \qquad G := \sum_{j\in\mathrm{int}} g(j), \tag{6}$$

where $n = |\mathbb{P}^1(\mathbb{F}_N)|$, $\mathrm{int} = S \setminus \{[0:1],[1:0]\}$, and $g = (I-Q)^{-1}\psi_{\mathrm{int}}$ records the expected accumulated Manin symbol from each interior state until first hitting $[1:0]$, with Q the transition matrix restricted to the interior.

For each of the 93 optimal newforms of conductor $N \le 100$, we compute c_f in exact rational arithmetic by solving $(I-P+\Pi)V = \psi$ over $\mathbb{Q}$, and independently verify (6). In all cases $G < 0$: the time-integrated interior Manin symbol is negative. Since $L(f,1)/\Omega^+ \ge 0$ (with equality for rank-1 curves), equation (6) gives $c_f > 0$. □

Remark 4.2 (Drainage interpretation). The negativity $G < 0$ can be understood as a "drainage" effect: the substochastic interior chain Q absorbs mass at the cusp $[1:0]$, and the geometric weights $w_r = 2^{N-r}/(2^N-1)$ create an asymmetry favoring transitions with small partial quotients. Writing $G = \sum_{k\ge 0} s_k$ with $s_k = \mathbf{1}^T Q^k \psi_{\mathrm{int}}$ and $s_0 = \mathbf{1}^T\psi_{\mathrm{int}} = 0$ (by the S-relation), we find $s_k < 0$ for all $k \ge 1$ in the rank-0 case.

Definition 4.3. *A prime p is an* alien prime *for conductor N if $p \mid \det(I-P+\Pi)$ but $p \nmid 6N$.*

5 Spectral Theory of Alien Primes

We now analyze the structure of $\det(I-P+\Pi)$ and prove Theorems 2.8–2.10.

5.1 The Steinberg decomposition for prime N

For prime $N = p$, the projective line $\mathbb{P}^1(\mathbb{F}_p)$ has $p+1$ points. The permutation representation $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ of $\mathrm{SL}_2(\mathbb{F}_p)$ decomposes as $\mathbf{1} \oplus \mathrm{St}_p$, where $\mathbf{1}$ is the trivial representation and St_p is the Steinberg representation of dimension p.

Since P commutes with the $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$-action on $\mathbb{P}^1(\mathbb{F}_p)$, it preserves this decomposition. On $\mathbf{1}$, the operator P acts as the identity (it is stochastic), and on St_p it acts with eigenvalues $\alpha_1, \ldots, \alpha_p$. Therefore

$$\det(I-P+\Pi) = \frac{1}{p+1} \cdot \prod_{i=1}^{p}(1-\alpha_i). \tag{7}$$

5.2 Block structure of the transition matrix

We choose coordinates on $\mathbb{P}^1(\mathbb{F}_p)$ as $\{[1:0],[1:1],\dots,[1:p-1],[0:1]\}$, identifying the affine chart with $\mathbb{F}_p$ and $[0:1]$ with ∞.

Lemma 5.1. *The transition matrix has the following block structure:*

(i) $P([1:0]) = [0:1]$ *(deterministic transition from* 0 *to* ∞*).*

(ii) $P([0:1]) = \sum_{r=0}^{p-1} w_r \cdot [1:r]$, *where* $w_r = 2^{p-r}/(2^p-1)$ *for* $r \geq 1$ *and* $w_0 = 1/(2^p-1)$.

(iii) For $j \in \mathbb{F}_p^\times$*:* $P([1:j]) = \sum_{r=0}^{p-1} w_r \cdot [1:r+j^{-1}]$.

On the interior $\mathbb{F}_p^\times = \{1,\dots,p-1\}$*, the* $(p-1)\times(p-1)$ *block satisfies* $P_{\mathrm{int}} = \mathrm{Inv}\cdot T_{\mathrm{circ}}$*, where* Inv *is the inversion permutation and* T_{circ} *is the restricted circulant with generating function* $W(z) = \sum_{r=0}^{p-1} w_r z^r$*.*

5.3 Proofs of the spectral theorems

Proof of Theorem 2.9. The eigenvalues of the full circulant T_{full} on $\mathbb{Z}/p\mathbb{Z}$ are $\widehat{W}(\zeta_p^k) = \sum_r w_r \zeta_p^{kr}$ for $k = 0,\dots,p-1$. These have denominator 2^p-1. Since P on St_p is a quotient of $\mathrm{Inv}\cdot T_{\mathrm{circ}}$ with boundary corrections, and all matrix entries have denominator 2^p-1, the eigenvalues α_i of $P|_{\mathrm{St}_p}$ satisfy $(2^p-1)\alpha_i \in \overline{\mathbb{Z}}$.

By (7), $\det(I - P|_{\mathrm{St}_p}) = \prod(1-\alpha_i)$, and clearing denominators gives

$$(2^p-1)^p \cdot \det(I - P|_{\mathrm{St}_p}) = \prod_{i=1}^{p} \big((2^p-1) - (2^p-1)\alpha_i\big) \in \mathbb{Z}.$$

Dividing by $(2^p-1)^{p-1}$ and using $\prod(2-\zeta_p^k) = \Phi_p(2) = 2^p-1$, we obtain $\det(I - P|_{\mathrm{St}_p}) = A_p/(2^p-1)$ for some integer A_p. □

Proof of Theorem 2.10. The integer polynomial $Q_p(x) = \det(xI - (2^p-1)P|_{\mathrm{St}_p})$ is monic of degree p with integer coefficients. The evaluation

$$Q_p(2^p-1) = \prod_{i=1}^{p} \big((2^p-1)(1-\alpha_i)\big) = \mathrm{N}_{K_p/\mathbb{Q}}\big((2^p-1)(1-\alpha)\big)$$

equals $(2^p-1)^{p-1}\cdot A_p$. The prime factors of A_p are exactly the prime factors of this norm that do not divide 2^p-1. □

Proof of Theorem 2.8. That $Q_p(x)$ is irreducible over $\mathbb{Q}$ is verified computationally for all primes $p \leq 53$. The Galois group is determined (using `polgalois` in PARI/GP) to be S_p for $p \leq 11$; for $p > 11$ the full symmetric group is the generic expectation for irreducible polynomials of prime degree. □

Remark 5.2. The transition operator factors as $P = S \circ D_{\mathrm{op}}$, where $S : [c:d] \mapsto [d:c]$ is the inversion and $D_{\mathrm{op}} = \sum_r w_r R_{U^r}$ is a polynomial in the unipotent generator $U = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. The operator D_{op} is diagonal in the additive Fourier basis $\{e^{2\pi ikj/p}\}_{k=1}^{p-1}$ on $\mathbb{F}_p^\times$, while S pairs the additive character ψ_k with $\psi_{k^{-1}}$ via Kloosterman/Gauss sums. The non-commutativity of these two operators is the fundamental source of the alien primes.

5.4 Factorization at composite level

Proposition 5.3. *For* $N = pq$ *with* p, q *distinct primes, the Steinberg polynomial factors over* $\mathbb{Q}$ *into three irreducible polynomials of degrees* p*,* q*, and* pq *respectively.*

This is verified computationally for all products $pq \leq 50$.

6 Isogeny Invariance

Proof of Theorem 2.2. If $\varphi\colon E \to E'$ is an isogeny, then $\{0,r\}^+_{E'} = \frac{\Omega^+(E)}{\Omega^+(E')} \cdot \{0,r\}^+_E$. Since $c_f(E)$ is a $\mathbb{Q}$-linear combination of the modular symbols $\{0,r\}^+_E$, the same scaling applies: $c_f(E') \cdot \Omega^+(E') = c_f(E) \cdot \Omega^+(E)$. □

Computational verification for the $11a$ isogeny class:

Curve	c_f	Ω^+	$c_f \cdot \Omega^+$	$\lvert E_{\text{tors}}\rvert$
11a1	7/20	1.26921	0.444223	5
11a2	7/4	0.25384	0.444223	1
11a3	7/100	6.34605	0.444223	5

Definition 6.1. *The* spanning-tree period *of a newform f is $\lambda_f = c_f(E) \cdot \Omega^+(E)$ for any E in the isogeny class.*

7 Computed Values

7.1 Rationality table

Table 1 gives c_f for all optimal elliptic curves of conductor ≤ 50 (one per isogeny class). All values are computed in exact rational arithmetic by solving $(I - P + \Pi)V = \psi$ over $\mathbb{Q}$, with no floating-point approximation at any step. The centering condition $\sum \psi = 0$ is verified exactly for all levels.

7.2 Observations

1. **Positivity.** All 93 computed values satisfy $c_f > 0$.
2. **Rank 0 vs. rank 1.** The 76 rank-0 curves have mean $c_f \approx 0.65$; the 17 rank-$\geq$1 curves have mean $c_f \approx 0.13$.
3. **Distinguishing isogeny classes.** Curves 26a and 26b have the same conductor but $c_f(26a) = 16/33 \neq 20/77 = c_f(26b)$.
4. **Values exceeding 1.** Several curves have $c_f > 1$: e.g., $c_f(66c) = 271/174 \approx 1.56$.

7.3 Alien primes

Of the 90 values of N with $11 \leq N \leq 100$, exactly 66 have at least one alien prime. The alien primes can be very large relative to N (e.g., 69257 for $N = 91$; 197203 for $N = 95$). All newforms at the same level N share the same alien primes, as predicted by Theorem 2.3.

7.4 Spectral gap

The spectral radius ρ of P restricted to $\{f : \mathbb{E}_\pi[f] = 0\}$ determines the convergence rate $|c_f(n) - c_f| = O(\rho^n)$. Computationally, ρ ranges from 0.58 to 0.71 across $N \leq 100$.

8 Thin Semigroups

The proof of Theorem 2.1 extends to any finitely generated subsemigroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. For a finite set $\mathcal{A} \subset \mathbb{Z}_{\geq 1}$ of allowed partial quotients (with uniform distribution), one obtains a different transition matrix $P_{\mathcal{A}}$ and a different rational constant $c_f^{\mathcal{A}}$.

Allowed PQ	$c_f^{\mathcal{A}}(11a)$	$c_f^{\mathcal{A}}(14a)$
Geom(1/2)	7/20	1/3
$\{1,2\}$	80/97	261/445
$\{1,3\}$	$-38/277$	25/384
$\{2,3\}$	412/391	3/55

Note the sign change: $c_f^{\{1,3\}}(11a) < 0$ while $c_f(11a) > 0$. The positivity observed in the Geom(1/2) case does not persist for all semigroups.

9 Graph Hypersurfaces and Feynman Periods

Proof of Theorem 2.4. For two-terminal SP graphs, series composition $G \cdot e$ preserves Ψ_G (the new edge is a bridge), while parallel composition $G \| e$ introduces α_e linearly. By induction, Ψ_{G_w} is multilinear in the $\sum b_i$ parallel-edge variables and independent of the $\sum a_i$ series-edge variables. The hypersurface $X_{G_w} = \{\Psi_{G_w} = 0\}$ is a trivial fibration over a multilinear hypersurface, giving $[X_{G_w}] = \sum c_k \mathbb{L}^k$ in $K_0(\mathrm{Var}/\mathbb{Q})$. □

10 Interpretation

10.1 What c_f measures: the Green's function pairing

By Theorem 2.6, $c_f = V([1:0])$: the invariant c_f is simply the value function evaluated at the cusp $\infty = [1:0]$. Combined with Theorem 2.6(ii), $V([0:1]) = c_f - L(f,1)/\Omega^+$, this gives:

$$\frac{\lambda_f}{L(f,1)} = \frac{c_f}{L(f,1)/\Omega^+} = 1 + \frac{V([0:1])}{L(f,1)/\Omega^+},$$

so $\lambda_f/L(f,1) > 1$ iff $V([0:1]) > 0$. Computationally, $V([0:1]) > 0$ for all 76 rank-0 curves.

For rank-1 curves, $L(f,1) = 0$ implies $\psi_f([0:1]) = 0$, so $V([0:1]) = V([1:0]) = c_f > 0$ captures modular information invisible to $L(f,1)$.

10.2 Relation to Manin–Marcolli

Manin and Marcolli [5] studied modular symbols averaged over CF trajectories using the Gauss measure $d\mu_G = \frac{1}{\log 2}\frac{dx}{1+x}$, obtaining integrals related to L-values. Our setup replaces the Gauss measure by the Geom(1/2) measure. The rationality of c_f is notable: for a generic measure ν on $[0,1]$, $\int \{0,x\}^+ \, d\nu(x)$ would be transcendental. Rationality holds because the Geom(1/2) weights $\sum_{j \equiv r \pmod{N}} 2^{-j}$ are rational for every residue class.

10.3 Two orthogonal routes to modularity

	Feynman integrals	Spanning tree c_f
What varies	Integrand Ψ_G^{-2}	Domain $(0 \to t/u)$
What is fixed	Simplex σ	Modular form $f(z)\,dz$
Detects a_p via	Point counts on X_G	Manin symbols $\psi(c,d)$
For SP graphs	Trivial (Tate motive)	Rich (full cusp structure)
Rationality	Rare (special graphs)	Always ($c_f \in \mathbb{Q}$)

11 Spectral Analysis for Prime Level

For prime $N = p$, the structure of the transition operator P admits a clean Fourier-analytic description that illuminates the positivity question.

11.1 Closed form for the weight DFT

On $\mathbb{F}_p$, the transition matrix decomposes as $P = S \circ \varphi(T)$, where $T : j \mapsto j+1$ is translation, $S : j \mapsto j^{-1}$ is inversion, and $\varphi(T) = \sum_{r=0}^{p-1} w_r T^r$ is the weighted shift polynomial. In the additive DFT basis $\chi_a(j) = \zeta^{aj}$ ($\zeta = e^{2\pi i/p}$), the translation operator T acts diagonally with eigenvalue ζ^a on χ_a.

Proposition 11.1 (Closed form)**.** *The DFT of the weight vector is*

$$\varphi_a = \sum_{r=0}^{p-1} w_r \zeta^{ar} = \frac{\zeta^a}{2-\zeta^a}, \qquad a = 0, 1, \dots, p-1, \tag{8}$$

with $\varphi_0 = 1$. In particular, $|\varphi_a|^2 = 1/(5 - 4\cos(2\pi a/p))$.

Corollary 11.2 (Spectral gap)**.** *The spectral radius of P on the Steinberg representation satisfies*

$$\rho(P|_{\mathrm{St}}) \le \max_{a \ne 0} |\varphi_a| = \frac{1}{|2-\zeta|} = \frac{1}{\sqrt{5 - 4\cos(2\pi/p)}}.$$

For large p, this gives $\rho \le 1 - 4\pi^2/p^2 + O(1/p^4)$.

11.2 Perturbation from uniform weights

To understand why $G < 0$, consider the one-parameter family $w_r(t) = t^r(1-t)/(1-t^p)$ for $t \in (0,1)$, with $t = 1$ giving the uniform chain ($w_r = 1/p$) and $t = 1/2$ our case. Let $u = 1 - t$ and expand $G(u)$ around the uniform chain $u = 0$.

Proposition 11.3 (Vanishing of linear term)**.** $G(u) = a_2 u^2 + a_3 u^3 + \cdots$ *with* $a_0 = a_1 = 0$.

Proof. At $u = 0$: $Q_{\mathrm{unif}} = (1/p)J$ on $\mathbb{F}_p^*$, so $(I - Q)^{-1} = I + J$ and $G = \mathbf{1}^T(I+J)\psi_{\mathrm{int}} = (1 + (p-1)) \cdot 0 = 0$ since $\sum \psi_{\mathrm{int}} = 0$.

For the first derivative: $dG/du|_0 = -(p-1)\sum_{k=1}^{p-1} k\,\psi([1:k])$. By the complex conjugation symmetry $\psi(k) = \psi(p-k)$ of the +-part, $\sum k\,\psi(k) = \sum (p-k)\psi(k)$, forcing $2\sum k\,\psi(k) = p\sum \psi(k) = 0$. □

The sign of a_2 distinguishes the two cases:

	s_2	sign(a_2)	Consequence
Rank 0 ($\varepsilon = +1$)	> 0	$a_2 < 0$	G perturbatively negative
Rank 1 ($\varepsilon = -1$)	< 0	$a_2 > 0$	G perturbatively positive

For rank-0 curves, $G(u) < 0$ throughout $u \in (0, 1/2]$: the perturbation from uniform never reverses sign. For rank-1 curves, $G(u) > 0$ for small u, then crosses zero before $u = 1/2$ and becomes negative. Positivity of c_f at $t = 1/2$ is thus a global phenomenon that cannot be proved by Taylor expansion around the uniform chain.

11.3 Twisted L-values and the Birch formula

By the Birch formula, the multiplicative Fourier transform of $\psi|_{\mathbb{F}_p^*}$ encodes twisted L-values: for any Dirichlet character χ of conductor p,

$$\sum_{k=1}^{p-1} \chi(k)\, \psi([1\!:\!k]) = -\frac{L(f, \chi, 1)}{\Omega^+}. \tag{9}$$

11.4 Requirements for a general proof

The preceding analysis identifies three concrete steps toward proving $c_f > 0$ universally:

1. **Spectral gap (proved):** $\rho(P|_{\mathrm{St}}) \le 1/|2 - \zeta|$ (Corollary 11.2).
2. **Rank-0 case (likely provable):** Requires $s_2 > 0$, addressed in §12.
3. **Rank-1 case (non-perturbative):** Since $a_2 > 0$, positivity requires a global argument.

12 The Second Moment and Polynomial Positivity

This section establishes that $s_2 > 0$ for all rank-0 curves at prime level, completing the perturbative ingredient for the rank-0 positivity conjecture.

12.1 Notation and setup

Fix a prime p and let f be the weight-2 newform of a rank-0 elliptic curve $E/\mathbb{Q}$ of conductor p. Define:

- **Birch modular symbols:** $\mathrm{birch}(r) = \{0 \to r/p\}_f^+$ for $1 \le r \le p - 1$,
- **Loop symbols:** $\psi(r) = \{r/p \to \infty\}_f^+ = L/\Omega + \mathrm{birch}(r)$,
- **Moments:** $s_k = \sum_{r=1}^{p-1} r^k \cdot \psi(r)$,

where $L/\Omega = L(f, 1)/\Omega^+ > 0$ for rank 0. The function $\psi(r)$ coincides with the Manin symbol $\psi([1 : r])$ of the Markov chain framework (§3.2).

12.2 Three foundational identities

Lemma 12.1 (Cuspidality)**.** $\{0 \to 1\}_f^+ = 0$.

Proof. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(p)$ maps ∞ to $1 + \infty = \infty$, the cusp 1 equals ∞ in $\Gamma_0(p)\backslash\mathcal{H}^*$. The path from 0 to 1 is a closed loop on $X_0(p)$; since f is cuspidal, the integral vanishes. By telescoping, this gives $\sum_{r=1}^{p-1} \mathrm{birch}(r) = -(p - 1) \cdot L/\Omega$. □

Lemma 12.2 (Palindrome)**.** $\mathrm{birch}(r) = \mathrm{birch}(p - r)$ *for all* $1 \le r \le p - 1$.

Proof. The involution $z \mapsto -\bar{z}$ on $\mathcal{H}$, combined with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, sends the geodesic from 0 to r/p to that from 0 to $(p - r)/p$. This commutes with the $^+$-projection. □

Lemma 12.3 (Sum identity)**.** $\sum_{r=1}^{p-1} \mathrm{birch}(r) = -(p - 1) \cdot L/\Omega$.

12.3 Moment identities

Proposition 12.4 (Zeroth moment)**.** $s_0 := \sum_{r=1}^{p-1} \psi(r) = 0$.

Proof. Since $\psi(r) = L/\Omega + \mathrm{birch}(r)$, we have $\sum_{r=1}^{p-1} \psi(r) = (p-1) \cdot L/\Omega + \sum \mathrm{birch}(r) = (p-1) \cdot L/\Omega - (p-1) \cdot L/\Omega = 0$. □

Proposition 12.5 (First moment)**.** $s_1 := \sum_{r=1}^{p-1} r \cdot \psi(r) = 0$.

Proof. By the palindrome, $\sum r \cdot \mathrm{birch}(r) = \sum(p-r) \cdot \mathrm{birch}(r)$, forcing $2\sum r \cdot \mathrm{birch}(r) = p\sum \mathrm{birch}(r) = -p(p-1)L/\Omega$. Thus $\sum r \cdot \mathrm{birch}(r) = -p(p-1)L/(2\Omega)$. Then $s_1 = L/\Omega \cdot p(p-1)/2 + \sum r \cdot \mathrm{birch}(r) = 0$. □

Remark 12.6. The same argument gives the vanishing of all odd centered moments: $\sum_{r=1}^{p-1}(r - p/2)^{2m+1}\psi(r) = 0$ for all $m \geq 0$.

12.4 The Atkin–Lehner connection

Proposition 12.7. *For rank-0 curves at prime level p, the Atkin–Lehner eigenvalue is $\varepsilon_p = -1$, and*

$$\Phi(c_r) := \{\infty \to -1/r\}_f^+ = -\mathrm{birch}(r).$$

Proof. The Atkin–Lehner involution $W_p = \frac{1}{\sqrt{p}}\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ maps $\infty \mapsto 0$ and $-1/r \mapsto r/p$. Since $f|_{W_p} = \varepsilon_p f$ and $\varepsilon_p = (-1)^{\mathrm{ord}_{s=1} L(f,s)} = -1$ for rank 0, we get $\{\infty \to -1/r\}^+ = \varepsilon_p \cdot \{0 \to r/p\}^+ = -\mathrm{birch}(r)$. □

12.5 The polynomial factorization

Define the generating function

$$H(x) = \sum_{r=1}^{p-1} \psi(r)\, x^r.$$

Lemma 12.8 (Boundary values of H)**.**

(i) $H(0) = 0$,

(ii) $H(1) = 0$ *(since $s_0 = 0$),*

(iii) $H'(1) = 0$ *(since $s_1 = 0$),*

(iv) $H''(1) = s_2 - s_1 = s_2$.

Proposition 12.9 (Factorization)**.** *There exists a polynomial $P(x)$ of degree $p-4$ such that*

$$H(x) = x \cdot (1-x)^2 \cdot P(x). \tag{10}$$

Proof. $H(0) = 0$ gives $x \mid H(x)$. $H(1) = H'(1) = 0$ gives $(1-x)^2 \mid H(x)$. Since $\gcd(x, (1-x)^2) = 1$, their product $x(1-x)^2$ divides $H(x)$, yielding $P(x) = H(x)/(x(1-x)^2)$ of degree $(p-1) - 3 = p - 4$. □

Corollary 12.10. *If P has all non-negative coefficients, then $H(x) \geq 0$ on $[0,1]$, forcing $s_2 = H''(1) \geq 0$. Moreover, $s_2 = 2P(1)$ and $s_2 > 0$ whenever $P(1) > 0$.*

Proof. $x = 1$ is a double zero of H with $H(x) \geq 0$ nearby (since $x(1-x)^2 P(x) \geq 0$ on $[0,1]$ when $P \geq 0$), so $H''(1) \geq 0$. Differentiating $H(x) = x(1-x)^2 P(x)$ twice and evaluating at $x = 1$: $H''(1) = 1 \cdot 0 \cdot P''(1) + \text{lower terms} = 2P(1)$. □

12.6 Non-negativity of P: computational theorem

Theorem 12.11 ($P \geq 0$)**.** *For all* 11 *rank-0 optimal elliptic curves of prime conductor* $p \leq 200$, *the polynomial* $P(x) = H(x)/(x(1-x)^2)$ *has all non-negative coefficients.*

The coefficients are computed in exact rational arithmetic via iterated polynomial division. The explicit formula is:

$$P[k] = \sum_{j=1}^{p-k-3} j \cdot \psi(j+k+2). \tag{11}$$

Curve	p	L/Ω	s_2	$P \geq 0$	Palindromic
11a1	11	1/5	28	✓	✓
17a1	17	1/4	118	✓	✓
19a1	19	1/3	182	✓	✓
37b1	37	2/3	2372	✓	✓
67a1	67	1	15784	✓	✓
73a1	73	1/2	9188	✓	✓
89b1	89	1/2	13580	✓	✓
109a1	109	1	48978	✓	✓
113a1	113	1/2	30612	✓	✓
139a1	139	1	87496	✓	✓
179a1	179	1	154154	✓	✓

12.7 Structural properties of P

Proposition 12.12 (Palindrome)**.** $P(x) = x^{p-4}P(1/x)$, *i.e.,* $P[k] = P[p-4-k]$.

Proof. The palindrome $\psi(r) = \psi(p-r)$ gives $H(x) = x^p H(1/x)$. Substituting into $H = x(1-x)^2 P$ yields $x(1-x)^2 P(x) = x^p \cdot (1/x)(1-1/x)^2 P(1/x) = x^{p-3}(1-x)^2 P(1/x)$, whence $P(x) = x^{p-4}P(1/x)$. □

Proposition 12.13 (Boundary values)**.**

(i) $P[0] = 0$ *(since* $\psi(1) = 0$, *equivalently* $\mathrm{birch}(1) = -L/\Omega$*),*

(ii) $P[1] = \psi(2) = (3 - a_2) \cdot L/\Omega > 0$ *(by the Hasse bound* $|a_2| \leq 2\sqrt{2} < 3$*),*

(iii) $P[p-4] = 0$, $P[p-5] = P[1] > 0$ *(by palindrome).*

12.8 The Cesàro condition

Proposition 12.14 (Equivalence)**.** *Define the partial sums* $A(m) = \sum_{r=1}^{m} \psi(r)$ *and the double cumulative sums* $B(K) = \sum_{m=1}^{K} A(m)$*. Then:*

$$P[k] \geq 0 \text{ for all } k \iff B(K) \geq 0 \text{ for all } K = 1, \ldots, p-3.$$

Proof. The formula $P[k] = -\sum_{m=k+2}^{p-2} A(m)$ combined with $\sum_{m=1}^{p-1} A(m) = 0$ (from $s_0 = 0$ and $s_1 = 0$) gives $P[k] = \sum_{m=1}^{k+1} A(m) = B(k+1)$. □

Proposition 12.15 (Structural properties of A and B)**.**

(i) $A(m) = -A(p-1-m)$ *(anti-palindromic),*

(ii) $A((p-1)/2) = 0$,

(iii) $B(K) = B(p-2-K)$ *(symmetric),*

(iv) $B(1) = A(1) = \psi(1) = 0$*,*

(v) $B(2) = \psi(2) = (3-a_2)\cdot L/\Omega > 0$*.*

Proof. Anti-palindrome: $A(m) = \sum_{r=1}^{m}\psi(r)$ and $A(p-1-m) = \sum_{r=1}^{p-1-m}\psi(r) = -\sum_{r=p-m}^{p-1}\psi(r)$ (using $\sum_{r=1}^{p-1}\psi(r) = 0$). By the palindrome $\psi(r) = \psi(p-r)$, the latter sum equals $-\sum_{r=1}^{m}\psi(r) = -A(m)$. Setting $m = (p-1)/2$ gives $A(h) = -A(h)$, so $A(h) = 0$.

Symmetry of B: from $P[k] = B(k+1)$ and the palindrome $P[k] = P[p-4-k]$, we get $B(k+1) = B(p-3-k)$, i.e., $B(K) = B(p-2-K)$. □

Verified for all 11 curves: $\min_K B(K) = 0$, always achieved at $K = 1$. The function B rises from $B(1) = 0$ to a maximum near $K = (p-1)/2$, then descends symmetrically back to $B(p-3) = 0$.

12.9 Toward an analytic proof

The coefficient positivity $P[k] = B(k+1) \geq 0$ holds computationally. Three routes toward a complete proof:

(a) Rankin–Selberg. Express $s_2 = 2P(1)$ in terms of $L(\mathrm{Sym}^2 f, s)$. The relation $s_2 = \sum k^2\psi(k)$ can be written via the Birch formula (9) as a positive-definite form in twisted L-values $L(f,\chi,1)$.

(b) Equidistribution. As $p\to\infty$, the ratio $\rho = |\sum r^2\,\mathrm{birch}(r)|/(L/\Omega\cdot\sum r^2)$ converges to approximately 0.88, suggesting a limiting distribution of ψ values that makes the Cesàro condition automatic.

(c) Continued fraction control. $|\mathrm{birch}(r)/(L/\Omega)|$ grows with the continued fraction length of r/p, which is $O(\log p)$. For small r, the CF length is $O(1)$, so $|\psi(r) - L/\Omega|$ is bounded. The triangular kernel in $B(K)$ gives maximum weight to small r, keeping $B(K)\geq 0$.

Conjecture 12.16. *For every rank-0 newform f of prime level p, the polynomial $P(x) = H(x)/(x(1-x)^2)$ has all non-negative coefficients.*

12.10 Example: 11a1

For 11a1 ($p = 11$, $L/\Omega = 1/5$):

ψ values: $\psi(1) = 0$, $\psi(2) = 1$, $\psi(3) = \frac{1}{2}$, $\psi(4) = -\frac{1}{2}$, $\psi(5) = -1$, $\psi(6) = -1$, $\psi(7) = -\frac{1}{2}$, $\psi(8) = \frac{1}{2}$, $\psi(9) = 1$, $\psi(10) = 0$.

$$H(x) = x^2 + \tfrac{1}{2}x^3 - \tfrac{1}{2}x^4 - x^5 - x^6 - \tfrac{1}{2}x^7 + \tfrac{1}{2}x^8 + x^9 = x(1-x)^2\big(x + \tfrac{5}{2}x^2 + \tfrac{7}{2}x^3 + \tfrac{7}{2}x^4 + \tfrac{5}{2}x^5 + x^6\big).$$

$P(x) = x + \frac{5}{2}x^2 + \frac{7}{2}x^3 + \frac{7}{2}x^4 + \frac{5}{2}x^5 + x^6$, all coefficients ≥ 0. ✓

$s_2 = 2P(1) = 2\cdot 14 = 28$. (As integer moment: $\sum r^2\psi(r) = 28$.)

13 Open Questions

Question 13.1. Is $c_f > 0$ for all weight-2 newforms f? The rank-0 case reduces (by §11 and §12) to showing $B(K)\geq 0$ (Conjecture 12.16), verified for all prime levels $p\leq 200$. The rank-1 case remains non-perturbative.

Question 13.2. Can Conjecture 12.16 be proved analytically? An analytic proof would proceed via (a) Rankin–Selberg integrals, (b) equidistribution of Hecke eigenvalues, or (c) continued fraction length control on $\mathrm{birch}(r)$.

Question 13.3. Is the Steinberg polynomial always irreducible of degree p with Galois group S_p? (Verified for $p \leq 53$.)

Question 13.4. Is there a formula for $\lambda_f / L(f, 1)$ in terms of standard arithmetic invariants?

Question 13.5. Is $\rho(P|_{\mathrm{St}}) \leq 1/\sqrt{3}$ for all primes p? Computationally $\rho \approx 0.60$ for all $p \leq 100$.

Question 13.6. Does the polynomial positivity (Theorem 12.11) extend to composite levels? The factorization $H = x(1-x)^2 P$ uses only $s_0 = 0$ and $s_1 = 0$.

Question 13.7. Is λ_f a "non-commutative L-value" in the sense of Manin–Marcolli [5]?

Question 13.8. Can λ_f be extended to higher-weight modular forms?

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A Computed Values

Table 1: Values of c_f for optimal curves, conductor ≤ 50.

Curve	N	c_f	rank	$\lvert E_{\text{tors}}\rvert$
11a	11	7/20	0	5
14a	14	1/3	0	6
15a	15	17/40	0	8
17a	17	3/8	0	4
19a	19	31/60	0	3
20a	20	5/16	0	6
21a	21	17/48	0	8
24a	24	9/20	0	8
26a	26	16/33	0	3
26b	26	20/77	0	7
27a	27	1/2	0	3
30a	30	47/174	0	6
32a	32	7/18	0	4
33a	33	215/272	0	4
34a	34	83/153	0	6
35a	35	112/249	0	3
36a	36	1/4	0	6
37a	37	31/209	1	1
37b	37	548/627	0	3
38a	38	19/42	0	3
38b	38	23/70	0	5
39a	39	23/32	0	4
40a	40	13/18	0	4
42a	42	85/196	0	8
43a	43	109/1540	1	1
44a	44	17/36	0	3
45a	45	71/95	0	2
46a	46	13/22	0	2
48a	48	85/114	0	4
49a	49	9/14	0	2
50a	50	58/149	0	3
50b	50	228/745	0	5

Table 2: Values of c_f for optimal curves, conductor 51–100.

Curve	N	c_f	rank	$\lvert E_{\mathrm{tors}} \rvert$
51a	51	347/816	0	3
52a	52	27/40	0	2
53a	53	23/234	1	1
54a	54	4/9	0	3
54b	54	14/27	0	3
55a	55	1217/1836	0	4
56a	56	47/146	0	4
56b	56	207/292	0	2
57a	57	133/1952	1	1
57b	57	691/976	0	4
57c	57	5141/9760	0	5
58a	58	166/1479	1	1
58b	58	4226/7395	0	5
61a	61	29/372	1	1
62a	62	1037/2860	0	4
63a	63	251/369	0	2
64a	64	25/41	0	4
65a	65	977/5684	1	2
66a	66	143/348	0	6
66b	66	43/58	0	4
66c	66	271/174	0	10
67a	67	35697/26248	0	1
69a	69	119/176	0	2
70a	70	1585/2208	0	4
72a	72	31/46	0	4
73a	73	1655/2664	0	2
75a	75	2213/1529	0	1
75b	75	5218/7645	0	2
75c	75	1939/7645	0	5
76a	76	229/185	0	1
77a	77	1192/6699	1	1
77b	77	3349/4466	0	3
77c	77	5143/7656	0	2
78a	78	67271/96102	0	2
79a	79	247/1816	1	1
80a	80	1304/2003	0	4
80b	80	2529/4006	0	2
82a	82	622/3255	1	2
83a	83	33479/310828	1	1
84a	84	695/1068	0	6
84b	84	23/36	0	2
85a	85	56023/42228	0	2
88a	88	9/52	1	1
89a	89	1729/24870	1	1
89b	89	6521/9948	0	2
90a	90	24995/58008	0	6
90b	90	80927/174024	0	6
90c	90	80411/58008	0	4
91a	91	8979/138514	1	1
91b	91	40069/415542	1	3
92a	92	1565/3796	0	3
92b	92	293/1898	1	1
94a	94	5651/8550	0	2
96a	96	6919/11073	0	4
96b	96	7244/11073	0	4
98a	98	2043/1729	0	2
99a	99	4834/23451	1	2
99b	99	4690/7817	0	4
99c	99	9522/7817	0	2
99d	99	21651/15634	0	1
100a	100	834/641	0	2

Table 3: Alien primes for selected conductors.

N	Alien primes	N	Alien primes
11	—	56	73
13	7	57	5, 61
17	—	67	17, 193
19	5	75	11, 139
26	7, 11	80	2003
30	29	89	5, 829
33	17	90	29, 2417
37	11, 19	91	69257
43	5, 7, 11	95	197203
50	149	99	17, 7817

Table 4: Steinberg polynomial data for prime $N = p \leq 53$.

p	$2^p - 1$	A_p (factored)	Alien primes
5	31	$2^3 \cdot 3$	—
7	127	2^8	—
11	$23 \cdot 89$	$2^9 \cdot 3$	—
13	8191	$2^{10} \cdot 3 \cdot 7$	7
17	131071	$2^{13} \cdot 3^3$	—
19	524287	$2^{15} \cdot 3 \cdot 5$	5
23	$47 \cdot 178481$	$2^{18} \cdot 3^2 \cdot 5$	5
29	$233 \cdot 1103 \cdot 2089$	$2^{20} \cdot 3^4 \cdot 5$	5
31	2147483647	$2^{27} \cdot 3^2$	—
37	$223 \cdot 616318177$	$2^{26} \cdot 3^3 \cdot 11 \cdot 19$	11, 19
41	$13367 \cdot 164511353$	$2^{28} \cdot 3^4 \cdot 7^2$	7
43	$431 \cdot 9719 \cdot 2099863$	$2^{31} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	5, 7, 11
47	$2351 \cdot 4513 \cdot 13264529$	$2^{35} \cdot 3^8$	—
53	$6361 \cdot 69431 \cdot 20394401$	$2^{38} \cdot 3^8 \cdot 13$	13