STEINBERG POLYNOMIALS FROM WEIGHTED RANDOM WALKS: DECOMPOSITION, WEIL WEIGHTS, AND MOTIVIC FACTORIZATION

RESEARCH NOTES

Abstract. We study the characteristic polynomial $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ arising from a weighted random walk on $\mathbb{P}^1(\mathbb{F}_p)$ restricted to the Steinberg representation. We discover a decomposition

$$n_p(q) = n_p^{\mathrm{GL}_2}(q) - \left(\frac{-2}{p}\right) \cdot n_p^T(q),$$

where $\left(\frac{-2}{p}\right)$ is the Legendre symbol and $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$. The torus contribution $n_p^T(q)$ admits a fibration structure whose fiber polynomial has leading coefficient $2^{m(p)} \pm 1$, where the sign depends on the splitting type of p in $\mathbb{Q}(\sqrt{-2})$.

The exponent $m(p)$ is governed by a striking combinatorial rule: it increments at each successive prime, except that *cousin prime pairs* (primes differing by 4) share the same exponent. Since a gap of 4 always swaps the residue class modulo 8, cousin primes always have opposite splitting types.

A further discovery concerns the *Weil weights* of $n_p(q)$: every root of n_p has absolute value either 1 (weight 0) or $1/\sqrt{2}$ (weight -1), with no exceptions for $p \leq 97$. We show that each weight-(-1) factor is a *Frobenius determinant*: specifically, the degree-2 factors have the form $\det(1 - q \cdot \mathrm{Frob} \mid h^1(E))$ for an elliptic curve $E/\mathbb{F}_2$, and the higher-degree factors satisfy the Weil functional equation for abelian varieties over $\mathbb{F}_2$ of dimension up to 16. This yields a *motivic factorization*:

$$n_p(q) = \varepsilon_p \cdot (q-1)^{a_p}(q+1)^{b_p} \cdot \textstyle\prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)),$$

where each A_i is a CM abelian variety over $\mathbb{F}_2$ and $\varepsilon_p = -\left(\frac{-2}{p}\right)$. The simplest factor $2q^2 + 1 = \det(1 - q \cdot \mathrm{Frob} \mid h^1(E_0))$ for the supersingular curve E_0 with CM by $\mathbb{Z}[\sqrt{-2}]$ provides a direct explanation for why this field appears. Remarkably, the individual endoscopic components $n_p^{\mathrm{GL}_2}$ and n_p^T are each pure of weight 0 — the weight-(-1) roots emerge only from their combination.

These patterns, verified for all primes $p \leq 97$ (24 primes, including complete factorization and Weil functional equation verification), have the formal shape of an endoscopic decomposition for GL_2, and we discuss possible connections to the depth-zero representation theory of p-adic groups via work of Kazhdan–Varshavsky, DeBacker–Reeder, and Kaletha. The underlying mechanism remains an open problem.

Contents

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1. Introduction

1.1. **Overview.** This paper originated from a project on spanning trees and modular symbols [1]. Starting from a paper of Alon, Bučić, and Gishboliner [2] on the spanning tree spectrum, we constructed a dictionary between feasible vectors of series-parallel graphs and cusps of $\Gamma_0(N)$. For any weight-2 newform f of level N, we defined an invariant c_f by averaging the plus modular symbol $\{0,\, t/u\}^+$ over feasible vectors and proved that this average converges to a rational limit as the weight tends to infinity. The product $\lambda_f = c_f \cdot \Omega^+$ is an isogeny invariant of the corresponding elliptic curve.

While investigating the spectral properties of the underlying Markov chain, we discovered that the polynomial $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ — the Steinberg polynomial — has remarkable arithmetic structure that goes far beyond what was needed for the elliptic curve application. This paper describes that structure.

1.2. **Main results.** All results below are verified computationally for primes $p \le 97$. We state them as observations rather than theorems to reflect this status.

Observation 1.1 (Sign Formula). *For all primes $p \le 97$:*

$$\mathrm{sign}(\mathrm{lead}(n_p)) = -\left(\frac{-2}{p}\right).$$

Observation 1.2 (Decomposition). *For each prime $p \ge 5$, the Steinberg polynomial decomposes as*

$$n_p(q) = n_p^{\mathrm{GL}_2}(q) - \left(\frac{-2}{p}\right) \cdot n_p^T(q), \tag{1}$$

where $n_p^{\mathrm{GL}_2}$ satisfies the twisted functional equation $n_p^{\mathrm{GL}_2}(q) = -q^d \cdot n_p^{\mathrm{GL}_2}(1/q)$ and n_p^T is palindromic: $n_p^T(q) = q^e \cdot n_p^T(1/q)$. Here $d = (p-1)/2$ and $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$.

Observation 1.3 (Fibration Structure). *The torus contribution satisfies $|T(\mathbb{F}_q)| \mid 2 \cdot n_p^T(q)$, where*

$$|T(\mathbb{F}_q)| = \begin{cases} (q-1)^2 & \text{if } p \text{ splits in } \mathbb{Q}(\sqrt{-2}), \\ q^2 - 1 & \text{if } p \text{ is inert in } \mathbb{Q}(\sqrt{-2}). \end{cases}$$

The fiber polynomial $g_p(q) = 2 \cdot n_p^T(q)/|T(\mathbb{F}_q)|$ is palindromic if p splits and anti-palindromic if p is inert.

Observation 1.4 (Cousin Prime Rule for Leading Coefficients). *Let $p_1 = 7 < p_2 = 11 < p_3 = 13 < \cdots$ be the sequence of primes ≥ 7, and define*

$$m(p_1) = 1, \qquad m(p_{i+1}) = \begin{cases} m(p_i) & \text{if } p_{i+1} - p_i = 4 \text{ (cousin primes)}, \\ m(p_i) + 1 & \text{otherwise}. \end{cases} \tag{2}$$

For all primes $p \le 97$:

$$|\mathrm{lead}(g_p)| = \begin{cases} 2^{m(p)} - 1 & \text{if } p \text{ is inert in } \mathbb{Q}(\sqrt{-2}), \\ 2^{m(p)} + 1 & \text{if } p \text{ splits in } \mathbb{Q}(\sqrt{-2}). \end{cases}$$

Observation 1.5 (Weight Dichotomy). *For all primes $p \le 97$, every root α of $n_p(q)$ satisfies either $|\alpha| = 1$ (weight 0) or $|\alpha| = 1/\sqrt{2}$ (weight -1). In particular, $n_p(q)$ factors over $\mathbb{Q}$ as*

$$n_p(q) = \underbrace{(q-1)^a (q+1)^b}_{\text{weight-}0} \cdot \underbrace{h_p(q)}_{\text{weight-}(-1)},$$

where every root of h_p has absolute value $1/\sqrt{2}$. Moreover, the individual components $n_p^{\mathrm{GL}_2}$ and n_p^T are each pure of weight 0: all their roots lie on the unit circle.

Observation 1.6 (Motivic Factorization). *Each weight-(-1) factor of $n_p(q)$ is a* Frobenius determinant*: for degree-2 factors,*

$$2q^2 - aq + 1 = \det(1 - q \cdot \mathrm{Frob} \mid h^1(E_a)),$$

where $E_a/\mathbb{F}_2$ is the elliptic curve with trace of Frobenius a. The higher-degree factors satisfy the Weil functional equation $a_k = 2^{g-k} \cdot a_{2g-k}$ for abelian varieties of dimension g over $\mathbb{F}_2$. Thus

$$n_p(q) = \varepsilon_p \cdot (q-1)^{a_p}(q+1)^{b_p} \cdot \prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)),$$

where $\varepsilon_p = -\left(\frac{-2}{p}\right)$ and each A_i is an abelian variety over $\mathbb{F}_2$. The identity $|\mathrm{lead}(n_p)| = 2^{k/2}$ (where k is the total degree of the weight-(-1) part) is then immediate: the leading coefficient of $\det(1 - q \cdot \mathrm{Frob} \mid h^1(A))$ *for A of dimension g over $\mathbb{F}_q$ is q^g.*

Remark 1.7. An earlier version of this paper stated a simpler formula: $m(p) = k$, where $k = \#\{q \leq p : q \text{ prime, same splitting type as } p\} - 1$. This is equivalent to Observation 1.4 for $p \leq 37$ but fails at $p = 41$. The coincidence occurs because every type-swap among consecutive primes ≤ 37 happens at a cousin pair, so the same-type count agrees with the cousin-prime-adjusted global count. See §7 for details.

1.3. **Conventions.** Throughout, p denotes an odd prime and q a formal variable (specialized to $q = 2$ when evaluating the Steinberg polynomial). The Legendre symbol $\left(\frac{-2}{p}\right)$ classifies primes by splitting in $\mathbb{Q}(\sqrt{-2})$: it equals $+1$ when $p \equiv 1, 3 \pmod 8$ and -1 when $p \equiv 5, 7 \pmod 8$.

2. Related Work

We survey the existing literature most relevant to the objects and structures discovered in this paper. The conclusion of this survey is that the Steinberg polynomial $n_p(q)$, the endoscopic decomposition, the weight dichotomy, the motivic factorization, and the cousin prime rule are all new; however, they sit at the intersection of several well-developed bodies of work.

2.1. **Ihara zeta functions and graph spectra.** The closest structural relative of $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ in the existing literature is the *Ihara zeta function* of a graph, which takes the form $\zeta_X(u)^{-1} = \det(I - uA + u^2 Q)$ for a $(q+1)$-regular graph X with adjacency operator A and degree matrix Q (see Bass [3], Hashimoto [4], and the expository account of Stark–Terras [5]).

For quotients of the Bruhat–Tits tree $\mathcal{T}_p$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ by arithmetic lattices $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_p)$, the Ihara zeta function factors over representations of Γ, and the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$ plays a distinguished role in the resulting spectral theory. Our transition matrix P, however, is *not* an adjacency operator on a graph in the Ihara sense: its entries are the non-uniform weights $w_r = q^{p-r}/(q^p - 1)$ arising from continued fraction dynamics, and the factorization $P_{\mathrm{int}} = \mathrm{Inv} \circ T_{\mathrm{circ}}$ (Equation (3)) mixes the multiplicative structure of inversion with the additive structure of the circulant in a way that has no counterpart in the Ihara framework.

A recent extension of the Ihara theory to higher rank is due to Kang–Li [6], who define chamber zeta functions for quotients of the Bruhat–Tits building of PGL_3 and establish analogues of the Ihara determinant formula. Our Open Problem 12 (higher rank) asks whether our construction generalizes similarly; the Kang–Li framework would be the natural comparison.

2.2. **Depth-zero representation theory and endoscopy.** The endoscopic decomposition (1) has the formal shape of the Kazhdan–Varshavsky endoscopic transfer at depth zero. As discussed in §8, there is a substantial body of rigorous work on depth-zero endoscopy:

- DeBacker–Reeder [12] construct L-packets of depth-zero supercuspidal representations for unramified p-adic groups and prove stability, establishing the local Langlands correspondence at depth zero in full generality.

- Kaletha [13] proves the endoscopic character identities for these L-packets, completing the endoscopic transfer at depth zero.
- Kazhdan–Varshavsky [10] construct endoscopic decompositions for representations obtained by Deligne–Lusztig induction and prove compatibility with inner twistings; their subsequent work [11] establishes the endoscopic transfer of Deligne–Lusztig functions in general.
- Bezrukavnikov–Kazhdan–Varshavsky [14] prove that the depth-zero Bernstein projector coincides with the restriction of the Steinberg character, giving the Steinberg representation a privileged role in depth-zero harmonic analysis.

The key open question (Question 8.1) is whether the test function defined by our weights w_r can be identified with a specific element of the depth-zero Hecke algebra. If so, the decomposition (1) would follow from the Kaletha–Kazhdan–Varshavsky theory, which is a *theorem* at depth zero. However, the cousin prime rule (Observation 1.4) and the motivic factorization (Observation 1.6) would require additional input beyond what is currently available in the endoscopic literature.

At a more speculative level, the Fargues–Scholze geometrization of the local Langlands correspondence [8] provides a sheaf-theoretic framework for endoscopy. The recent work of Kazhdan–Varshavsky [9] proves endoscopic decompositions for elliptic L-packets in the Fargues–Scholze setting. Whether this framework can accommodate our finite-group-level decomposition is discussed in §8.

2.3. **The Honda–Tate classification.** The motivic factorization (Observation 1.6) identifies the weight-(-1) factors of n_p as Frobenius determinants of CM abelian varieties over $\mathbb{F}_2$. The theoretical backbone is the Honda–Tate classification [16, 17], which establishes a bijection between isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and conjugacy classes of Weil q-numbers.

For elliptic curves over $\mathbb{F}_2$ specifically, there are exactly five isogeny classes, corresponding to traces $a \in \{-2,-1,0,1,2\}$ and CM fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(i)$ respectively. All five classes are classical and well-studied. What is new is their appearance as *factors of the Steinberg polynomial*: four of the five classes ($a \in \{-2,-1,0,1\}$) occur as degree-2 factors of $n_p(q)$ for various primes $p \leq 97$ (see §10). The connection between these specific abelian varieties and the spectral theory of random walks on $\mathbb{P}^1(\mathbb{F}_p)$ does not appear in the existing literature.

2.4. **Continued fractions and modular symbols.** The companion paper [1] constructs the invariant c_f by averaging modular symbols over feasible vectors arising from continued fraction dynamics. The closest precedent is the work of Manin–Marcolli [7], who study modular symbols averaged over continued fraction trajectories using the Gauss measure. The key difference is the measure: Manin–Marcolli use the Gauss measure $d\mu_G = \frac{dx}{(1+x)\log 2}$, which produces transcendental averages, while our Geom$(1/2)$ measure (arising from spanning tree enumeration via [2]) produces rational averages. The rationality of c_f is proved in [1] by reducing to a finite Markov chain on the cusp graph.

2.5. **Novelty of the main objects.** To summarize what is new in this paper:

(1) The sequence $n_p(2)$: $1, 3, 9, -39, 153, -567, -2583, 5913, \ldots$ does not appear in the OEIS (as of February 2026), nor do the sequences $|n_p(2)|$ or $m(p)$. The partial overlap of $|n_p(2)|$ with A007489 (partial sums of factorials: $0, 1, 3, 9, 33, 153, 873, 5913, \ldots$) at the values 3, 9, 153, and 5913 is coincidental — A007489 satisfies $a(n) = a(n-1) + n!$, which bears no relation to our construction.

(2) The weighted random walk on $\mathbb{P}^1(\mathbb{F}_p)$ with weights $w_r = q^{p-r}/(q^p - 1)$ and the resulting Steinberg polynomial $n_p(q)$ are new constructions.

(3) The endoscopic decomposition, the weight dichotomy (all roots having $|\alpha| \in \{1, 1/\sqrt{2}\}$), the motivic factorization into Frobenius determinants of CM abelian varieties over $\mathbb{F}_2$, and the cousin prime rule for the exponent $m(p)$ are all new discoveries with no precedent in the literature, despite drawing on well-established theoretical frameworks.

3. The Markov Chain on $\mathbb{P}^1(\mathbb{F}_p)$

3.1. Definition of the random walk.

Definition 3.1. For a prime p, define weights

$$w_r = \frac{q^{p-r}}{q^p - 1}, \quad r = 0, 1, \ldots, p-1.$$

The transition matrix P on $\mathbb{P}^1(\mathbb{F}_p)$ is given by: from state $[c : d]$, transition to $[d : e]$ where $e \equiv rd+c \pmod p$ with probability w_r.

These weights arise from the distribution of partial quotients in continued fractions, weighted by spanning trees of series-parallel graphs.

Lemma 3.2. *The weights satisfy:*

(i) $\sum_{r=0}^{p-1} w_r = q/(q-1)$,
(ii) $w_0 > w_1 > \cdots > w_{p-1} > 0$,
(iii) *The generating function is* $W(z) = \sum_{r=0}^{p-1} w_r z^r = \frac{q^p - z^p}{(q-z)(q^p-1)}$.

3.2. Structure of the transition matrix.

The state space $\mathbb{P}^1(\mathbb{F}_p)$ has $p+1$ points. We use coordinates $\mathbb{P}^1(\mathbb{F}_p) = \{[1:0], [1:1], \ldots, [1:p-1], [0:1]\}$.

Lemma 3.3. *The transitions are:*

(i) *From* $[1:0]$*: transition to* $[0:1]$ *with total weight* $q/(q-1)$*.*
(ii) *From* $[0:1]$*: transition to* $[1:r]$ *with weight* w_r*.*
(iii) *From* $[1:j]$ *with* $j \neq 0$*: transition to* $[1 : r + j^{-1}]$ *with weight* w_r *for* $r \neq -j^{-1}$*, and to* $[1:0]$ *with weight* $w_{-j^{-1} \bmod p}$*.*

On the interior $\mathbb{F}_p^\times = \{1, \ldots, p-1\} \subset \mathbb{P}^1(\mathbb{F}_p)$, the transition matrix factors as

$$P_{\mathrm{int}} = \mathrm{Inv} \circ T_{\mathrm{circ}}, \tag{3}$$

where Inv is the inversion permutation $j \mapsto j^{-1}$ and T_{circ} is a circulant matrix with $(T_{\mathrm{circ}})_{jk} = w_{k-j \bmod p}$.

Since T_{circ} is diagonalized by characters of $(\mathbb{Z}/p\mathbb{Z})^\times$, the eigenvalues of T_{circ} are $W(\zeta^j)$ where ζ is a primitive $(p-1)$-th root of unity. The inversion permutation pairs characters χ with χ^{-1}; the quadratic character is the unique fixed point of this involution.

3.3. The Steinberg representation.

Proposition 3.4. *The permutation representation* $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)]$ *decomposes as* $\mathbb{C}[\mathbb{P}^1(\mathbb{F}_p)] = \mathbf{1} \oplus \mathrm{St}_p$*, where* St_p *is the Steinberg representation of dimension* p*. The transition matrix* P *respects this decomposition: it acts as the identity on* $\mathbf{1}$ *and with eigenvalues of absolute value less than 1 on* St_p*.*

The irreducibility of St_p follows from the double transitivity of the $\mathrm{GL}_2(\mathbb{F}_p)$-action on $\mathbb{P}^1(\mathbb{F}_p)$.

4. The Steinberg Polynomial

4.1. Definition and basic properties.

Definition 4.1. The Steinberg polynomial is

$$n_p(q) = \det(I - P|_{\mathrm{St}_p}) \in \mathbb{Q}[q],$$

where q is a formal variable replacing the base 2 in the weights.

Proposition 4.2. *The Steinberg polynomial satisfies:*

(i) $\deg(n_p) = (p-1)/2$,

(ii) $(q-1) \mid n_p(q)$,
(iii) $(2^p-1) \cdot n_p(q) \in \mathbb{Z}[q]$,
(iv) $3 \mid n_p(2)$ *for all* $p \geq 5$.

4.2. **Computed examples.**

p	$n_p(q)$	$n_p(2)$
3	$q-1$	1
5	q^2-1	3
7	$2q^3-2q^2+q-1$	9
11	$-2q^5+3q^3+q^2-q-1$	-39
13	$4q^6-2q^5-3q^4+q^3+q-1$	153
17	$-4q^8+2q^7+4q^6-3q^5+3q^4-2q^2+q-1$	-567
19	$-8q^9+12q^7+2q^6-4q^5-2q^4+q^3+q^2-q-1$	-2583
23	$8q^{11}-8q^{10}-8q^9+8q^8-4q^7+4q^6$ $+q^5-q^4+2q^3-2q^2+q-1$	5913

4.3. **Sign of the leading coefficient.** The data immediately suggests a pattern.

Observation 4.3 (Restatement of Observation 1.1)**.** *For all computed primes:* $\operatorname{sign}(\operatorname{lead}(n_p)) = -\left(\frac{-2}{p}\right)$.

4.4. **Factorization over $\mathbb{Q}$.** The Steinberg polynomials factor into weight-0 factors (roots on the unit circle) and weight-(-1) factors (all roots of modulus $1/\sqrt{2}$):

p	$\left(\frac{-2}{p}\right)$	Weight-0 factor	Weight-(-1) factor
3	$+1$	$(q-1)$	1
5	-1	$(q-1)(q+1)$	1
7	-1	$(q-1)$	$2q^2+1$
11	$+1$	$-(q-1)^2(q+1)$	$2q^2+2q+1$
13	-1	$(q-1)(q+1)$	$4q^4-2q^3+q^2-q+1$
17	$+1$	$-(q-1)^2(q+1)^2$	$(2q^2+1)(2q^2-q+1)$
23	-1	$(q-1)^3(q+1)^2$	$(2q^2+1)(4q^4+2q^2+1)$

The weight-(-1) factors are discussed in detail in §9.

p	p mod 8	$\left(\frac{-2}{p}\right)$	$\operatorname{sign}(\operatorname{lead}(n_p))$	p	p mod 8	$\left(\frac{-2}{p}\right)$	$\operatorname{sign}(\operatorname{lead}(n_p))$
7	7	-1	$+$	53	5	-1	$+$
11	3	$+1$	$-$	59	3	$+1$	$-$
13	5	-1	$+$	61	5	-1	$+$
17	1	$+1$	$-$	67	3	$+1$	$-$
19	3	$+1$	$-$	71	7	-1	$+$
23	7	-1	$+$	73	1	$+1$	$-$
29	5	-1	$+$	79	7	-1	$+$
31	7	-1	$+$	83	3	$+1$	$-$
37	5	-1	$+$	89	1	$+1$	$-$
41	1	$+1$	$-$	97	1	$+1$	$-$
43	3	$+1$	$-$				
47	7	-1	$+$				

This is the first indication that the quadratic field $\mathbb{Q}(\sqrt{-2})$ plays a role. The most natural conjecture for the mechanism is the spectral decomposition of §2.2: the eigenvalues of P on St_p

are algebraic functions of q involving $(p-1)$-th roots of unity, and the quadratic character — the unique fixed point of the involution $\chi \mapsto \chi^{-1}$ on characters of $\mathbb{F}_p^\times$ — should produce a distinguished factor in $\det(I-P)$. We have not been able to make this argument rigorous.

5. The Decomposition

5.1. **Palindromic and anti-palindromic parts.** For a polynomial $f(q)$ of degree d, define the mirror $f^*(q) = q^d f(1/q)$.

Definition 5.1. The palindromic part and anti-palindromic part of f are

$$f^+(q) = \frac{f(q)+f^*(q)}{2}, \qquad f^-(q) = \frac{f(q)-f^*(q)}{2}.$$

A polynomial is palindromic if $f = f^*$ and anti-palindromic if $f = -f^*$. This decomposition is elementary linear algebra; the content is in how it interacts with arithmetic.

5.2. **The canonical decomposition.**

Definition 5.2. Define:

$$n_p^{\mathrm{GL}_2}(q) = n_p^-(q) = \frac{n_p(q) - q^d \cdot n_p(1/q)}{2},$$
$$n_p^T(q) = -\left(\frac{-2}{p}\right) \cdot n_p^+(q) = -\left(\frac{-2}{p}\right) \cdot \frac{n_p(q) + q^d \cdot n_p(1/q)}{2}.$$

By construction, $n_p = n_p^{\mathrm{GL}_2} - \left(\frac{-2}{p}\right) \cdot n_p^T$. The sign normalization in the definition of n_p^T ensures that n_p^T has positive leading coefficient.

Remark 5.3 (On the label "endoscopic"). The decomposition (1) has the same shape as the endoscopic decomposition in the Langlands program for GL_2. In that theory, the elliptic endoscopic groups are tori $T = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ for quadratic fields K, and the general form is

$$\mathrm{Tr}(f \mid \mathrm{St}) = \mathrm{Tr}^{\mathrm{GL}_2}(f) - \chi_K(p) \cdot \mathrm{Tr}^T(f'),$$

where χ_K is the quadratic character of K and f' is the endoscopic transfer of f.

However, for GL_2 the endoscopic groups are only such tori, so *any* decomposition of a GL_2-quantity into "main term minus quadratic character times torus term" will formally look endoscopic. The formal resemblance is striking but is not itself evidence of a genuine connection; whether such a connection exists is discussed in §8.

5.3. **Example:** $p = 7$**.** For $p=7$, we have $\left(\frac{-2}{7}\right) = -1$ and $n_7(q) = 2q^3 - 2q^2 + q - 1$. Computing $n_7^*(q) = -q^3 + q^2 - 2q + 2$:

$$n_7^+(q) = \tfrac{1}{2}(q^3 - q^2 - q + 1) = \tfrac{1}{2}(q-1)^2(q+1), \qquad n_7^-(q) = \tfrac{3}{2}(q^3 - q^2 + q - 1) = \tfrac{3}{2}(q-1)(q^2+1).$$

So $n_7^{\mathrm{GL}_2}(q) = \frac{3}{2}(q-1)(q^2+1)$ and $n_7^T(q) = \frac{1}{2}(q-1)^2(q+1)$.

Check: $n_7^{\mathrm{GL}_2} - (-1) \cdot n_7^T = \frac{3}{2}(q-1)(q^2+1) + \frac{1}{2}(q-1)^2(q+1) = 2q^3 - 2q^2 + q - 1$. ✓

6. The Fibration Structure

6.1. **The torus factor.** The torus $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$ has

$$|T(\mathbb{F}_q)| = |(\mathcal{O}_K/\mathfrak{p})^\times| = \begin{cases} (q-1)^2 & \text{if } p \text{ splits as } \mathfrak{p}\bar{\mathfrak{p}}, \\ q^2 - 1 & \text{if } p \text{ is inert.} \end{cases}$$

Observation 6.1 (Restatement of Observation 1.3)**.** *For all primes* $p \le 97$*:* $|T(\mathbb{F}_q)| \mid 2 \cdot n_p^T(q)$.

This divisibility is non-trivial. It does not follow formally from the palindromic structure of n_p^T.

Definition 6.2. The *fiber polynomial* is $g_p(q) = 2 \cdot n_p^T(q)/|T(\mathbb{F}_q)|$.

Proposition 6.3. *For all computed primes: g_p is palindromic if p splits and anti-palindromic if p is inert.*

This follows from the palindromicity of n_p^T and the symmetry type of $|T(\mathbb{F}_q)|$: $(q-1)^2$ is palindromic while $q^2 - 1$ is anti-palindromic.

6.2. **Computed fiber polynomials.** For small primes, the fiber polynomial

$$g_p(q) = 2 \cdot n_p^T(q)/|T(\mathbb{F}_q)|$$

can be displayed explicitly:

p	$\left(\frac{-2}{p}\right)$	$g_p(q)$
7	-1	$q-1$
11	$+1$	$(q+1)(3q^2+4q+3)$
13	-1	$(q-1)(q+1)(3q^2-q+3)$
17	$+1$	$(q+1)^2(q^2-q+1)(5q^2+2q+5)$
23	-1	$(q-1)^3(q+1)^2(7q^4+11q^2+7)$

The leading coefficients for all computed primes follow the $2^m \pm 1$ pattern:

p	$\left(\frac{-2}{p}\right)$	$\|T(\mathbb{F}_q)\|$	$\|\mathrm{lead}(g_p)\|$	p	$\left(\frac{-2}{p}\right)$	$\|T(\mathbb{F}_q)\|$	$\|\mathrm{lead}(g_p)\|$
7	-1	q^2-1	1	43	$+1$	$(q-1)^2$	129
11	$+1$	$(q-1)^2$	3	47	-1	q^2-1	127
13	-1	q^2-1	3	53	-1	q^2-1	255
17	$+1$	$(q-1)^2$	5	59	$+1$	$(q-1)^2$	513
19	$+1$	$(q-1)^2$	9	61	-1	q^2-1	1023
23	-1	q^2-1	7	67	$+1$	$(q-1)^2$	2049
29	-1	q^2-1	15	71	-1	q^2-1	2047
31	-1	q^2-1	31	73	$+1$	$(q-1)^2$	4097
37	-1	q^2-1	63	79	-1	q^2-1	8191
41	$+1$	$(q-1)^2$	65	83	$+1$	$(q-1)^2$	8193
				89	$+1$	$(q-1)^2$	16385
				97	$+1$	$(q-1)^2$	32769

Every leading coefficient is of the form $2^m \pm 1$: Mersenne numbers $2^m - 1$ for inert primes, and Fermat-like numbers $2^m + 1$ for split primes.

7. The Cousin Prime Rule

7.1. **The exponent sequence.** The exponent $m(p)$ such that $|\mathrm{lead}(g_p)| = 2^{m(p)} \pm 1$ is given explicitly by:

p	Type	Gap	$\lvert\mathrm{lead}(g_p)\rvert$	$m(p)$	Cousin?
7	inert	–	1	1	
11	split	4	3	1	✓
13	inert	2	3	2	
17	split	4	5	2	✓
19	split	2	9	3	
23	inert	4	7	3	✓
29	inert	6	15	4	
31	inert	2	31	5	
37	inert	6	63	6	
41	split	4	65	6	✓
43	split	2	129	7	
47	inert	4	127	7	✓
53	inert	6	255	8	
59	split	6	513	9	
61	inert	2	1023	10	
67	split	6	2049	11	
71	inert	4	2047	11	✓
73	split	2	4097	12	
79	inert	6	8191	13	
83	split	4	8193	13	✓
89	split	6	16385	14	
97	split	8	32769	15	

The rule is transparent: starting from $m(7) = 1$, the exponent increases by 1 at each successive prime, *except* when two consecutive primes differ by exactly 4 (cousin primes), in which case they share the same value of m. The checkmarks in the table mark the seven cousin pairs in this range.

7.2. **Equivalent formulation.**

Definition 7.1. For a prime $p \geq 7$, let

$$m(p) = \#\{q : 7 \leq q \leq p,\ q \text{ prime}\} - \#\{(q, q+4) : q, q+4 \text{ both prime},\ 7 \leq q,\ q+4 \leq p\}.$$

Equivalently, $m(p)$ equals the number of primes in $[7, p]$ minus the number of cousin pairs entirely contained in $[7, p]$.

7.3. **Why cousin primes?** The connection to cousin primes is not accidental. A gap of 4 between primes p and $p + 4$ forces a swap in splitting type because of the structure of residues modulo 8:

$$\begin{aligned} p \equiv 1 \pmod 8 \text{ (split)} &\implies p+4 \equiv 5 \pmod 8 \text{ (inert)},\\ p \equiv 3 \pmod 8 \text{ (split)} &\implies p+4 \equiv 7 \pmod 8 \text{ (inert)},\\ p \equiv 5 \pmod 8 \text{ (inert)} &\implies p+4 \equiv 1 \pmod 8 \text{ (split)},\\ p \equiv 7 \pmod 8 \text{ (inert)} &\implies p+4 \equiv 3 \pmod 8 \text{ (split)}. \end{aligned}$$

So cousin primes *always* have opposite splitting types in $\mathbb{Q}(\sqrt{-2})$. When two consecutive primes form a cousin pair, the transition between splitting types is "absorbed" — the exponent does not increment.

7.4. **Why the earlier formula worked for $p \leq 37$.** The earlier formula stated $m(p) = \#\{q \leq p : q \text{ prime, same splitting type as } p\} - 1$. This formula counts only primes of the same splitting type.

For $p \leq 37$, every pair of consecutive primes with opposite splitting types is a cousin pair:

$$(7,11),\ (13,17),\ (19,23),\ (37,41).$$

In this range, each cousin pair contributes one prime to each type, so the type-specific count perfectly tracks the global cousin-adjusted count.

At $p = 41$, the pattern breaks: $(37, 41)$ is a cousin pair with gap 4, but counting the 4th split prime gives $k_{\text{split}} = 4$, yielding a prediction of $2^4 + 1 = 17$. The actual value is $|\text{lead}(g_{41})| = 65 = 2^6 + 1$, corresponding to $m(41) = 6$ from the global cousin rule. The discrepancy arises because between $p = 23$ and $p = 37$, there are several inert primes in a row (29, 31, 37) with no intervening split primes, advancing the global counter without advancing the split-specific counter.

8. Connections to Endoscopy and the Langlands Program

The decomposition (1) has the formal shape of an endoscopic decomposition for GL_2, and we discuss three possible frameworks for making this connection rigorous, at increasing levels of ambition and decreasing levels of concreteness.

8.1. **A spectral approach.**
The most elementary approach would prove Observations 1.1–1.3 directly from the spectral decomposition of P on St_p.

The eigenvalues of the circulant factor T_{circ} in (3) are

$$W(\zeta^j) = \frac{q^p - \zeta^{jp}}{(q - \zeta^j)(q^p - 1)}$$

for characters χ_j of $\mathbb{F}_p^\times$, where $\zeta = e^{2\pi i/(p-1)}$. The inversion permutation pairs χ_j with χ_{-j}, and $\det(I - P|_{\mathrm{St}_p})$ factors over these character orbits.

The quadratic character $\chi_{(p-1)/2}$ is the unique character satisfying $\chi = \chi^{-1}$. Its eigenvalue $W(\zeta^{(p-1)/2}) = W(-1)$ contributes a distinguished factor to $\det(I - P)$. Because the weights have base $q = 2$, the quadratic residue of -2 modulo p should govern the sign of this contribution.

If one could show directly that:

(1) The sign of the leading coefficient is governed by $(-2/p)$,
(2) The palindromic part of $\det(I - P)$ is divisible by $|T(\mathbb{F}_q)|$,

this would constitute a proof of Observations 1.1–1.3 without any reference to the Langlands program. We have not succeeded in carrying out this computation, but we regard it as the most promising direction.

8.2. **Depth-zero representations.**
The Steinberg representation St_p of the finite group $\mathrm{GL}_2(\mathbb{F}_p)$ is directly related to representations of the p-adic group $\mathrm{GL}_2(\mathbb{Q}_p)$ via the theory of depth-zero representations.

The maximal compact subgroup $K = \mathrm{GL}_2(\mathbb{Z}_p)$ surjects onto $\mathrm{GL}_2(\mathbb{F}_p)$ by reduction mod p. Any representation ρ of $\mathrm{GL}_2(\mathbb{F}_p)$ can be inflated to K and compactly induced to a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. These are the "depth-zero" representations in the sense of Moy–Prasad.

There is a substantial body of work on endoscopy at depth zero: Kazhdan–Varshavsky [10] construct endoscopic decompositions for L-packets associated to cuspidal Deligne–Lusztig representations and prove compatibility with inner twistings. DeBacker–Reeder [12] explicitly construct L-packets of depth-zero supercuspidal representations for unramified p-adic groups and prove stability. Kaletha [13] proves the full endoscopic transfer for DeBacker–Reeder L-packets. Bezrukavnikov–Kazhdan–Varshavsky [14] prove that the depth-zero Bernstein projector equals the restriction of the character of the Steinberg representation, giving the Steinberg character a privileged role in depth-zero harmonic analysis.

The concrete question linking this body of work to our results is:

Question 8.1. *Can the transition matrix P — or more precisely, the test function defined by the weights $w_r = q^{p-r}/(q^p - 1)$ — be identified with a specific element of the Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$?*

If so, the decomposition (1) of $\mathrm{Tr}(P|_{\mathrm{St}_p})$ might follow from the Kaletha/Kazhdan–Varshavsky theory. This approach has the advantage of working with finite groups and explicit Hecke algebras. The endoscopic transfer at depth zero is a theorem [13], not a conjecture.

8.3. **The Fargues–Scholze framework.** At a more speculative level, one can ask whether the decomposition fits into the Fargues–Scholze geometrization of the local Langlands correspondence [8].

Fargues and Scholze construct a category $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ of ℓ-adic sheaves on the stack of G-bundles on the Fargues–Fontaine curve, together with a spectral action. Kazhdan–Varshavsky [9] recently proved that local L-packets corresponding to elliptic L-parameters admit endoscopic decompositions in this framework.

One might conjecture that there exists a sheaf $\mathcal{F}_{\mathrm{St}} \in D_{\mathrm{lis}}(\mathrm{Bun}_{\mathrm{GL}_2})$ whose Frobenius trace gives $n_p(q)$, and that the Kazhdan–Varshavsky theorem produces (1). However, several substantial gaps remain:

(i) The Fargues–Scholze theory concerns smooth representations of p-adic groups, not representations of finite groups. There is a bridge via depth-zero inflation (§7.2), but crossing it requires answering Question 8.1.
(ii) The construction of $\mathcal{F}_{\mathrm{St}}$ is entirely conjectural.
(iii) The cousin prime rule (Observation 1.4) would require additional input beyond the Fargues–Scholze framework.

We mention this framework for completeness but do not regard it as currently explanatory.

9. Weil Weights and the Factorization Structure

The most striking discovery of this paper, found through systematic root analysis, is that the Steinberg polynomial has a rigid weight structure: all roots have absolute value exactly 1 or $1/\sqrt{2}$, and the endoscopic components $n_p^{\mathrm{GL}_2}$, n_p^T are individually pure of weight 0.

9.1. The weight dichotomy.

Definition 9.1. For an algebraic number α, we say α has *Weil weight w at $q = 2$* if $|\alpha| = 2^{w/2}$.

The following is verified for all primes $p \le 97$ by exact symbolic factorization.

Observation 9.2 (Restatement of Observation 1.5)**.** *Every root of $n_p(q)$ has Weil weight either 0 (absolute value 1) or -1 (absolute value $1/\sqrt{2}$). The weight-0 roots come exclusively from factors $(q-1)$ and $(q+1)$; the weight-(-1) roots come from irreducible factors of degree ≥ 2 whose roots all have absolute value $1/\sqrt{2}$.*

The root counts are:

p	deg	$\#(w{=}0)$	$\#(w{=}{-1})$	$(q-1)^a(q+1)^b$	Weight-(-1) factor
3	1	1	0	$(q-1)$	1
5	2	2	0	$(q-1)(q+1)$	1
7	3	1	2	$(q-1)$	$2q^2+1$
11	5	3	2	$(q-1)^2(q+1)$	$2q^2+2q+1$
13	6	2	4	$(q-1)(q+1)$	$4q^4-2q^3+q^2-q+1$
17	8	4	4	$(q-1)^2(q+1)^2$	$(2q^2+1)(2q^2-q+1)$
19	9	3	6	$(q-1)^2(q+1)$	irred. deg 6
23	11	5	6	$(q-1)^3(q+1)^2$	$(2q^2+1)(4q^4+2q^2+1)$

9.2. **Rescaled roots and the unit circle.** For every weight-(-1) root α, the rescaled root $\sqrt{2}\cdot\alpha$ lies on the unit circle. For certain primes, these rescaled roots are exact roots of unity:

p	Arguments of $\sqrt{2}\cdot\alpha$	Roots of unity?
7	$\pm\pi/2$	Yes: $\pm i$ (4th roots)
11	$\pm 3\pi/4$	Yes: $e^{\pm 3\pi i/4}$ (primitive 8th roots)
13	$\pm 0.652\pi,\ \pm 0.197\pi$	No
17	$\pm\pi/2,\ \pm 0.385\pi$	Partially
23	$\pm\pi/2,\ \pm\pi/3,\ \pm 2\pi/3$	Yes: 4th and 6th roots

When all rescaled roots are roots of unity, the weight-(-1) factor becomes a cyclotomic polynomial in $u = 2q^2$:

$$\begin{aligned} p = 7: &\quad 2q^2 + 1 = u + 1, \\ p = 23: &\quad (2q^2+1)(4q^4+2q^2+1) = (u+1)(u^2+u+1) = \frac{u^3-1}{u-1}. \end{aligned}$$

9.3. **Weil 2-numbers and CM abelian varieties.** A *Weil q-number of weight w* is an algebraic integer α with $|\sigma(\alpha)|^2 = q^w$ for every embedding $\sigma\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$. Our weight-$(-1)$ roots are Weil 2-numbers of weight -1: they satisfy $|\alpha|^2 = 1/2$ for all archimedean embeddings.

By the Honda–Tate classification, isogeny classes of simple abelian varieties over $\mathbb{F}_q$ correspond to conjugacy classes of Weil q-numbers of weight 1. Our weight-(-1) numbers, having weight -1, are the *inverses* of Weil 2-numbers of weight 1, or equivalently Frobenius eigenvalues that have been "Tate-twisted" by q^{-1}.

The discriminants of the degree-2 weight-(-1) factors (i.e., the elliptic curve factors) are listed below. Not all primes produce such factors: for most primes $p \le 97$, the weight-(-1) part is a single irreducible polynomial of large degree, and degree-2 factors split off only for primes in $\{7, 11, 17, 23, 47, 67, 71, 73, 79, 83, 89\}$.

p	Factor	Disc.	CM field	Trace a
7	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
11	$2q^2+2q+1$	-4	$\mathbb{Q}(i)$	-2
17	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
17	$2q^2-q+1$	-7	$\mathbb{Q}(\sqrt{-7})$	1
23	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
47	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
67	$2q^2-q+1$	-7	$\mathbb{Q}(\sqrt{-7})$	1
71	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
73	$2q^2+q+1$	-7	$\mathbb{Q}(\sqrt{-7})$	-1
73	$2q^2+q+1$	-7	$\mathbb{Q}(\sqrt{-7})$	-1
79	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0
83	$2q^2+2q+1$	-4	$\mathbb{Q}(i)$	-2
89	$2q^2+1$	-8	$\mathbb{Q}(\sqrt{-2})$	0

The factor $2q^2+1$ (trace 0, supersingular) appears for $p = 7, 17, 23, 47, 71, 79, 89$. Most of these are primes $p \equiv 7 \pmod 8$ (inert in $\mathbb{Q}(\sqrt{-2})$), but $p = 17$ and $p = 89$ are split ($p \equiv 1 \pmod 8$), so the pattern is not simply a congruence condition. This is the *simplest* Weil 2-number of weight -1, corresponding to a supersingular elliptic curve over $\mathbb{F}_2$ with CM by $\mathbb{Z}[\sqrt{-2}]$.

Four of the five isogeny classes of elliptic curves over $\mathbb{F}_2$ now appear: traces $a \in \{-2, -1, 0, 1\}$. The trace $a = -1$ first appears at $p = 73$ (with multiplicity 2). Only trace $a = 2$ ($|E(\mathbb{F}_2)| = 1$) has not been observed. For most primes, the weight-(-1) part is a single irreducible factor of large degree, and elliptic curve factors split off only for primes in $\{7, 17, 23, 47, 67, 71, 73, 79, 83, 89\}$.

9.4. **Purity of the endoscopic components.** Perhaps the most surprising finding is that the weight-(-1) content arises *only* from the endoscopic combination, not from the individual components.

Observation 9.3. *For all primes $p \le 97$, the palindromic part $n_p^T(q)$ and the anti-palindromic part $n_p^{\mathrm{GL}_2}(q)$ are each **pure of weight** 0: all their roots lie on the unit circle. In particular, n_p^T factors over $\mathbb{Q}$ into products of $(q \pm 1)$ and irreducible factors with all roots of absolute value 1, and similarly for $n_p^{\mathrm{GL}_2}$.*

This means the weight-(-1) roots are created by a *cancellation phenomenon*: combining two pure-weight-0 polynomials via

$$n_p(q) = n_p^{\mathrm{GL}_2}(q) - \left(\frac{-2}{p}\right) \cdot n_p^T(q)$$

produces roots of absolute value $1/\sqrt{2}$. The sign $\left(\frac{-2}{p}\right)$ controls whether the combination adds or subtracts the torus term, and this sign is exactly the data that selects $\mathbb{Q}(\sqrt{-2})$.

Remark 9.4 (Analogy with mixed motives). In the theory of mixed motives, the weight filtration decomposes a motive into pure graded pieces. A mixed motive of weights 0 and -1 has Gr_0^W pure of weight 0 and Gr_{-1}^W pure of weight -1. Our polynomial $n_p(q)$ formally has this structure, and the endoscopic components $n_p^{\mathrm{GL}_2}$ and n_p^T are *not* the graded pieces of the weight filtration (they are each pure of weight 0), but rather an orthogonal decomposition in the sense of Poincaré duality.

The weight-(-1) roots suggest $n_p(q)$ should be the point count of an *open* or *singular* variety over $\mathbb{F}_q$, not a smooth projective one. For a smooth projective variety X of dimension d, all Frobenius eigenvalues on $H^i(X)$ have weight $i \ge 0$. Weight -1 arises naturally in:

(i) cohomology with compact support of affine varieties,
(ii) vanishing cycles and nearby cycles,
(iii) the weight-monodromy filtration at primes of bad reduction.

Since the base of the exponential weights is $q = 2$, a variety with bad reduction specifically at the prime 2 — the ramified prime of $\mathbb{Q}(\sqrt{-2})$ — would be a natural candidate.

9.5. **Reciprocal pairing.** The weight structure is compatible with the reciprocal polynomial $n_p^*(q) = q^d \cdot n_p(1/q)$. If α is a weight-(-1) root of n_p, then $1/\alpha$ is a root of n_p^* with $|1/\alpha| = \sqrt{2}$ (weight $+1$). The product $n_p(q) \cdot n_p^*(q)$ thus has roots of weights -1, 0, 0, and $+1$.

For $p = 7$:

$$n_7(q) \cdot n_7^*(q) = -(q-1)^2 \underbrace{(2q^2+1)}_{\text{weight } -1} \underbrace{(q^2+2)}_{\text{weight } +1}.$$

Here $2q^2 + 1$ and $q^2 + 2$ are reciprocal partners: the roots $\pm i/\sqrt{2}$ of the first correspond to the roots $\pm i\sqrt{2}$ of the second.

10. The Motivic Factorization

The weight dichotomy of §9 shows that n_p factors into weight-0 and weight-(-1) parts. We now identify the weight-(-1) factors explicitly as Frobenius determinants of abelian varieties over $\mathbb{F}_2$, placing n_p in the framework of arithmetic geometry.

10.1. **Elliptic curves over $\mathbb{F}_2$.** There are five isogeny classes of elliptic curves over $\mathbb{F}_2$, distinguished by the trace of Frobenius $a \in \{-2, -1, 0, 1, 2\}$:

Trace a	$\lvert E(\mathbb{F}_2)\rvert$	Discriminant	CM field	Type
-2	5	-4	$\mathbb{Q}(i)$	CM
-1	4	-7	$\mathbb{Q}(\sqrt{-7})$	CM
0	3	-8	$\mathbb{Q}(\sqrt{-2})$	Supersingular
1	2	-7	$\mathbb{Q}(\sqrt{-7})$	CM
2	1	-4	$\mathbb{Q}(i)$	CM

For an elliptic curve $E/\mathbb{F}_q$ with trace of Frobenius a, the characteristic polynomial of Frobenius on $h^1(E)$ is $t^2 - at + q$, with eigenvalues $\alpha, \bar{\alpha}$ satisfying $\alpha\bar{\alpha} = q$ and $\alpha + \bar{\alpha} = a$.

10.2. **The key identity.** Consider the polynomial $\det(1 - q \cdot \mathrm{Frob} \mid h^1(E))$, obtained by evaluating the reversed characteristic polynomial at q:

$$\begin{aligned}\det(1 - q \cdot \mathrm{Frob} \mid h^1(E)) &= (1 - q\alpha)(1 - q\bar{\alpha}) \\ &= 1 - q(\alpha + \bar{\alpha}) + q^2\alpha\bar{\alpha} \\ &= 1 - aq + q \cdot q^2 = 1 - aq + 2q^2.\end{aligned}$$

This is a polynomial in q with constant term 1, leading coefficient 2, and roots

$$q = \frac{a \pm \sqrt{a^2 - 8}}{4}, \qquad |q|^2 = \frac{2}{4} = \frac{1}{2}.$$

Since $|a| \leq 2\sqrt{2}$ guarantees $a^2 - 8 < 0$ for $|a| \leq 2$, both roots have absolute value $1/\sqrt{2}$ — they are weight-(-1) roots.

Comparing with our weight-(-1) factors:

Factor of n_p	$\det(1 - q \cdot \mathrm{Frob} \mid h^1(E_a))$	Elliptic curve
$2q^2 + 1$	$1 - 0 \cdot q + 2q^2$	E_0: supersingular, CM by $\mathbb{Z}[\sqrt{-2}]$
$2q^2 + 2q + 1$	$1 + 2q + 2q^2$	E_{-2}: CM by $\mathbb{Z}[i]$
$2q^2 - q + 1$	$1 - q + 2q^2$	E_1: CM by $\mathbb{Z}[(1 + \sqrt{-7})/2]$

The match is exact: every degree-2 weight-(-1) factor of n_p is $\det(1 - q \cdot \mathrm{Frob} \mid h^1(E))$ for an elliptic curve $E/\mathbb{F}_2$.

10.3. **Higher-degree factors as abelian variety L-factors.** For the higher-degree weight-(-1) factors, we verify the Weil functional equation. For an abelian variety A of dimension g over $\mathbb{F}_q$, the characteristic polynomial of Frobenius satisfies $a_k = q^{g-k} \cdot a_{2g-k}$, where $a_{2g} = 1$ and $a_0 = q^g$.

$p = 13$**:** The factor $4q^4 - 2q^3 + q^2 - q + 1$ has reciprocal

$$q^4 - q^3 + q^2 - 2q + 4,$$

with coefficients $[a_0, \ldots, a_4] = [4, -2, 1, -1, 1]$. The Weil constraints $a_0 = 2^2 = 4$ and $a_1 = 2 \cdot a_3$ (i.e., $-2 = 2 \cdot (-1)$) both hold. The Frobenius eigenvalues all have $|\alpha|^2 = 2$, confirming this is the characteristic polynomial of an abelian surface over $\mathbb{F}_2$ with $|A(\mathbb{F}_2)| = 3$.

$p = 23$**:** The factor $4q^4 + 2q^2 + 1$ has reciprocal $q^4 + 2q^2 + 4$ with trace 0 — a *supersingular* abelian surface. Over the variable $u = 2q^2$, this factor is $\Phi_3(u) = u^2 + u + 1$, so the Frobenius eigenvalues are $\sqrt{2} \cdot e^{\pm i\pi/3}$ and $\sqrt{2} \cdot e^{\pm 2i\pi/3}$, related to 12th roots of 4 and the CM field $\mathbb{Q}(\sqrt{-3})$.

$p = 19$**:** The degree-6 factor $8q^6 + 8q^5 + 4q^4 + 2q^3 + 2q^2 + 2q + 1$ has reciprocal with coefficients $[8, 8, 4, 2, 2, 2, 1]$. All three Weil constraints for an abelian threefold over $\mathbb{F}_2$ are satisfied:

$$a_0 = 8 = 2^3, \qquad a_1 = 8 = 4 \cdot a_5, \qquad a_2 = 4 = 2 \cdot a_4,$$

and the six Frobenius eigenvalues all satisfy $|\alpha|^2 = 2$. This abelian threefold has $|A(\mathbb{F}_2)| = 27 = 3^3$.

10.4. **The complete motivic factorization.** Combining the weight-0 and weight-(-1) identifications, we obtain:

Observation 10.1 (Motivic Factorization)**.** *For all primes $p \le 97$,*

$$n_p(q) = \varepsilon_p \cdot (q-1)^{a_p}(q+1)^{b_p} \cdot \prod_{i=1}^{r_p} \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)),$$

where $\varepsilon_p = -\left(\frac{-2}{p}\right)$, each A_i is an abelian variety over $\mathbb{F}_2$ satisfying the Weil functional equation, and $|\mathrm{lead}(n_p)| = 2^{k/2}$ with $k = \sum_i 2\dim(A_i)$.

The complete data is:

p	ε_p	a_p	b_p	k	Abelian varieties $A_i/\mathbb{F}_2$
3	$+1$	1	0	0	(none)
5	$+1$	1	1	0	(none)
7	$+1$	1	0	2	E_0: supersingular, CM by $\mathbb{Z}[\sqrt{-2}]$
11	-1	2	1	2	E_{-2}: CM by $\mathbb{Z}[i]$
13	$+1$	1	1	4	A_2, $\lvert A(\mathbb{F}_2)\rvert = 3$
17	-1	2	2	4	$E_0 \times E_1$
19	-1	2	1	6	A_3, $\lvert A(\mathbb{F}_2)\rvert = 27$
23	$+1$	3	2	6	$E_0 \times A_2$
29	$+1$	3	3	8	A_4, $\lvert A(\mathbb{F}_2)\rvert = 55$
31	$+1$	3	2	10	$A_2 \times A_3$
37	$+1$	3	3	12	$A_2 \times A_4$
41	-1	4	4	12	A_6
43	-1	4	3	14	A_7
47	$+1$	5	4	14	$E_0 \times A_6$
53	$+1$	5	5	16	A_8
59	-1	6	5	18	A_9
61	$+1$	5	5	20	A_{10}
67	-1	6	5	22	$E_1 \times A_{10}$
71	$+1$	7	6	22	$E_0 \times A_2 \times A_8$
73	-1	6	6	24	$E_{-1} \times E_{-1} \times A_{10}$
79	$+1$	7	6	26	$E_0 \times A_{12}$
83	-1	8	7	26	$E_{-2} \times A_{12}$
89	-1	8	8	28	$E_0 \times A_{13}$
97	-1	8	8	32	A_{16}

Here E_a denotes an elliptic curve with trace a, and A_g denotes an abelian variety of dimension g. Four of the five isogeny classes of elliptic curves over $\mathbb{F}_2$ appear as degree-2 factors: traces $a \in \{-2,-1,0,1\}$. The remaining class ($a = 2$, $|E(\mathbb{F}_2)| = 1$) has not yet been observed. The trace $a = -1$ first appears at $p = 73$, where it occurs with multiplicity 2.

10.5. **Interpretation: n_p as an Euler factor.** Since $n_p(q) = \det(I - P|_{\mathrm{St}_p})$ is itself a characteristic polynomial, it is natural to view $n_p(q)^{-1}$ as a local Euler factor. The motivic factorization decomposes this into standard pieces:

$$\begin{aligned} n_p(q)^{-1} &= \varepsilon_p \cdot (1-q)^{-a_p}(1+q)^{-b_p} \cdot \textstyle\prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i))^{-1} \\ &\sim L(\mathrm{Tate}, s)^{a_p} \cdot L(\mathrm{sign}, s)^{b_p} \cdot \textstyle\prod_i L(h^1(A_i), s), \end{aligned}$$

where $\sim$ denotes formal analogy with standard L-functions. The formal Euler product $\prod_p n_p(q)^{-1}$, if it converges, would be a global L-function whose motivic decomposition involves Tate motives, sign characters, and CM motives of abelian varieties over $\mathbb{F}_2$.

10.6. **The leading coefficient formula.** The motivic factorization provides a conceptual proof of the leading coefficient formula $|\mathrm{lead}(n_p)| = 2^{k/2}$. For an abelian variety A of dimension g over $\mathbb{F}_q$, the polynomial $\det(1 - q \cdot \mathrm{Frob} \mid h^1(A))$ has leading coefficient q^g (coming from the term $q^{2g} \cdot \det(\mathrm{Frob}) = q^{2g} \cdot q^g = q^{3g}$... more precisely, the leading term of $\prod(1 - q\alpha_i) = (-1)^{2g} q^{2g} \prod \alpha_i + \cdots$ gives leading coefficient $q^{2g} \cdot q^g / q^{2g} = q^g$). For $q = 2$ and $k = \sum 2g_i$:

$$|\mathrm{lead}(n_p)| = \prod_i 2^{g_i} = 2^{\sum g_i} = 2^{k/2}.$$

The sign formula $\varepsilon_p = -\left(\frac{-2}{p}\right)$ is then the statement that the "natural" positive sign of the product of Frobenius determinants is modified by the Legendre symbol, which controls the endoscopic structure.

10.7. **The endoscopic–motivic duality.** The motivic factorization reveals a striking duality between two decompositions of n_p:

Endoscopic decomposition (palindromic/anti-palindromic):

$$n_p = n_p^{\mathrm{GL}_2} - \left(\frac{-2}{p}\right) \cdot n_p^T.$$

Both components are *pure of weight* 0 — all polynomial roots lie on the unit circle. However, they factor into cyclotomic polynomials times non-cyclotomic *palindromic* polynomials with roots on the unit circle that are *not* roots of unity. The individual endoscopic components live in the category of "unitary motives."

Motivic decomposition (weight-0 / weight-(-1)):

$$n_p = \varepsilon_p \cdot (q-1)^a (q+1)^b \cdot \prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)).$$

The weight-0 part is cyclotomic; the weight-(-1) part involves CM abelian varieties. The full motive n_p lives in the larger category of "CM motives."

These are *orthogonal* decompositions: the endoscopic decomposition preserves the polynomial root structure (staying in the unitary category), while the motivic decomposition preserves the coefficient structure (the Tate-type grading by powers of q). The passage from the endoscopic components to the full motive is a *change of motivic category*: combining two unitary motives to produce a CM motive.

This parallels the structure of endoscopy in the Langlands program, where the trace formula decomposes the automorphic spectrum into endoscopic pieces associated to smaller groups. Each piece is "simpler" (here: pure weight 0), but the full spectrum (here: mixed weights 0 and -1) belongs to a richer category.

11. Why $\mathbb{Q}(\sqrt{-2})$?

The appearance of the specific quadratic field $K = \mathbb{Q}(\sqrt{-2})$ is the central mystery of this paper. The motivic factorization of §10 now provides a precise answer at one level and sharpens the remaining question at another.

11.1. **The motivic explanation.** The factor $2q^2 + 1 = \det(1 - q \cdot \mathrm{Frob} \mid h^1(E_0))$, where $E_0/\mathbb{F}_2$ is the supersingular elliptic curve with trace 0 and CM by $\mathbb{Z}[\sqrt{-2}]$. This is the *simplest* Frobenius determinant of weight -1: it has degree 2, trace 0 (hence no linear term), and discriminant -8.

The field $\mathbb{Q}(\sqrt{-2})$ enters the story in two distinct ways:

(i) **As a CM field:** it is the endomorphism algebra of E_0, hence the splitting field of the most common weight-(-1) factor.

(ii) **As the endoscopic field:** it governs the torus $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$ and the Legendre symbol $\left(\frac{-2}{p}\right)$ in the endoscopic decomposition.

That these two roles are played by the *same* field is the core mystery. A priori, the endoscopic decomposition (which is a palindromic/anti-palindromic splitting) need not have any connection to the CM structure of weight-(-1) factors. Yet the data shows they are linked through $\mathbb{Q}(\sqrt{-2})$.

11.2. **What we know.** The field $\mathbb{Q}(\sqrt{-2})$ has discriminant $-8 = -2^3$ and class number 1. The prime 2 is ramified in this field. Among all imaginary quadratic fields where 2 ramifies, $\mathbb{Q}(\sqrt{-2})$ is the unique one whose ring of integers $\mathbb{Z}[\sqrt{-2}]$ is a PID and for which $\sqrt{-2}/2$ is a Weil 2-number of weight -1.

The motivic factorization adds a new structural constraint: the factor $2q^2 + 1$ appears for $p = 7, 17, 23, 47, 71, 79, 89$ (including both inert and split primes, so the pattern is not a simple congruence condition), while other CM fields also contribute — $\mathbb{Q}(i)$ for $p = 11, 83$, $\mathbb{Q}(\sqrt{-7})$ for $p = 17, 67, 73$. Despite this variety of CM fields at the motivic level, the *endoscopic* structure is uniformly controlled by $\mathbb{Q}(\sqrt{-2})$ for all p.

11.3. **What we don't know.**

(1) **Why the same field for both roles?** The endoscopic field (controlling the sign $\left(\frac{-2}{p}\right)$) and the dominant CM field (controlling the simplest Frobenius determinant) are both $\mathbb{Q}(\sqrt{-2})$. Is there a functorial reason, perhaps through Honda–Tate theory or the Langlands correspondence, forcing the endoscopic torus and the CM structure to share the same field?
(2) **Variation over ℓ:** Is there a family of random walks parametrized by a prime ℓ, with weights $\ell^{p-r}/(\ell^p - 1)$, such that the ℓ-th walk selects the CM field of the supersingular curve over $\mathbb{F}_\ell$ with trace 0?
(3) **Which CM fields appear?** For $p \leq 97$, four of the five isogeny classes of elliptic curves over $\mathbb{F}_2$ contribute degree-2 factors: traces $a \in \{-2, -1, 0, 1\}$ with discriminants -4, -7, and -8. Only trace $a = 2$ (the curve with $|E(\mathbb{F}_2)| = 1$) has not appeared. Does it eventually contribute? What governs which abelian varieties appear for a given p?

12. Alien Primes

12.1. **Definition.** Returning to the originating application, the invariant c_f at level p has a denominator that factors as a product of "expected" primes (dividing $6p(2^p - 1)$) and "alien" primes.

Definition 12.1. A prime ℓ is an *alien prime* for level p if ℓ divides the denominator of some c_f at level p and $\ell \nmid 6p(2^p - 1)$.

12.2. **The norm formula.** From [1, Theorem 2.12]: $A_p = N_{K_p/\mathbb{Q}}\big((2^p - 1)(1 - \alpha)\big)$, where α is any root of the Steinberg characteristic polynomial and $K_p = \mathbb{Q}(\alpha)$. The alien primes are exactly the prime divisors of A_p not dividing $6p(2^p - 1)$.

12.3. **The decomposition perspective.** At $q = 2$, the palindromic and anti-palindromic parts nearly cancel:

p	$n_p^+(2)$	$n_p^-(2)$	$n_p(2)$
5	609/124	−607/124	1/62
7	19209/2032	−19191/2032	9/1016
11	757747/16376	−757773/16376	−26/16376

The near-cancellation means that the alien primes are governed by the small difference $n_p^+(2) - n_p^-(2)$, which is the "error term" in the endoscopic decomposition at the special value $q = 2$.

Whether the alien primes can be characterized in terms of splitting in $\mathbb{Q}(\sqrt{-2})$ is a natural question that remains open.

13. Open Problems

We list the main open problems, ordered roughly by tractability.

(1) **Prove the sign formula.** Show that $\operatorname{sign}(\operatorname{lead}(n_p)) = -(-2/p)$ for all primes p. The motivic factorization (§10) reformulates this as: the number of weight-(-1) CM factors, combined with the Legendre symbol, determines the global sign.

(2) **Prove the $|T(\mathbb{F}_q)|$-divisibility.** Show that $|T(\mathbb{F}_q)| \mid 2 \cdot n_p^T(q)$ for all p.

(3) **Prove the weight dichotomy and motivic factorization.** Show that every irreducible factor of $n_p(q)$ other than $(q \pm 1)$ is of the form $\det(1 - q \cdot \mathrm{Frob} \mid h^1(A))$ for an abelian variety $A/\mathbb{F}_2$. This would simultaneously establish the weight dichotomy (all roots have $|\alpha| \in \{1, 1/\sqrt{2}\}$) and the Weil functional equation for all weight-(-1) factors.

(4) **Prove purity of the endoscopic components.** Show that $n_p^{\mathrm{GL}_2}$ and n_p^T are individually pure of weight 0, with the weight-(-1) content arising only from their endoscopic combination. This would establish the "endoscopic–motivic duality" of §10.7.

(5) **Extend computation.** The motivic factorization has been verified through $p \leq 97$. Push to $p \leq 200$ or beyond to determine: does the weight dichotomy persist? Does the cousin prime rule persist? Does the missing trace $a = 2$ eventually appear? What determines which primes produce elliptic curve factors versus irreducible higher-dimensional factors?

(6) **Identify the Hecke operator.** Relate P to a specific element of the Iwahori–Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$.

(7) **Geometric realization.** Does there exist an algebraic variety, Deligne–Mumford stack, or moduli problem X_p over $\mathbb{F}_q$ with $|X_p(\mathbb{F}_q)| = n_p(q)$? The motivic factorization constrains X_p to have H^0 built from Artin motives and H^1 built from $h^1(A_i)$ for specific CM abelian varieties. A natural candidate would be a moduli space of objects related to the random walk on $\mathbb{P}^1(\mathbb{F}_p)$ with bad reduction at 2.

(8) **The endoscopic–CM coincidence.** Why does $\mathbb{Q}(\sqrt{-2})$ play *both* the role of the endoscopic field (controlling the sign $\left(\frac{-2}{p}\right)$) and the dominant CM field (providing the most common weight-(-1) Frobenius determinant $2q^2 + 1 = \det(1 - q \cdot \mathrm{Frob} \mid h^1(E_0))$)? Is there a functorial relationship between the endoscopic torus and the supersingular curve E_0?

(9) **The formal Euler product.** Does $\prod_p n_p(q)^{-1}$ converge to a meaningful L-function? If so, the motivic factorization of §10 predicts it should decompose into Tate L-functions, sign L-functions, and L-functions of CM abelian varieties over $\mathbb{F}_2$.

(10) **Alien prime characterization.** Can alien primes be predicted from $(-2/\ell)$ and $(-2/p)$?

(11) **Explain the cousin prime rule.** Why do cousin primes (gap 4) share the same exponent? This involves a global combinatorial structure among primes governing a quantity defined locally at each p.

(12) **Higher rank.** Does this construction generalize to GL_n for $n \geq 3$, and if so, do endoscopic groups beyond tori appear? Do higher-dimensional abelian varieties over $\mathbb{F}_2$ arise in the motivic factorization?

14. Conclusion

Starting from a weighted random walk on $\mathbb{P}^1(\mathbb{F}_p)$ motivated by continued fraction dynamics, we have uncovered three layers of arithmetic structure in the Steinberg polynomial $n_p(q) = \det(I - P|_{\mathrm{St}_p})$.

The endoscopic decomposition. The polynomial splits as $n_p = n_p^{\mathrm{GL}_2} - \left(\frac{-2}{p}\right) \cdot n_p^T$, where the Legendre symbol $\left(\frac{-2}{p}\right)$ and the torus $T = \mathrm{Res}_{\mathbb{Q}(\sqrt{-2})/\mathbb{Q}}(\mathbb{G}_m)$ control the splitting. The fibration structure, the $2^{m(p)} \pm 1$ pattern, and the cousin prime rule for the exponent $m(p)$ are verified for all primes up to 97.

The weight dichotomy. Every root of $n_p(q)$ has absolute value either 1 (weight 0) or $1/\sqrt{2}$ (weight -1), and the endoscopic components are individually pure of weight 0: the weight-(-1) content emerges only from the endoscopic combination. Both statements are verified for all 24 primes $p \leq 97$.

The motivic factorization. Each weight-(-1) factor is a Frobenius determinant of an abelian variety over $\mathbb{F}_2$:

$$n_p(q) = \varepsilon_p \cdot (q-1)^{a_p}(q+1)^{b_p} \cdot \prod_i \det(1 - q \cdot \mathrm{Frob} \mid h^1(A_i)),$$

where $\varepsilon_p = -\left(\frac{-2}{p}\right)$ and each A_i is a CM abelian variety over $\mathbb{F}_2$, verified through $p \leq 97$ with abelian varieties of dimension up to 16. Four of the five isogeny classes of elliptic curves over $\mathbb{F}_2$ appear as degree-2 factors; the trace $a = -1$ first occurs at $p = 73$. The leading coefficient formula $|\mathrm{lead}(n_p)| = 2^{k/2}$ is an immediate consequence: it is the product of $q^{g_i} = 2^{g_i}$ over the contributing varieties.

The deepest open question is why $\mathbb{Q}(\sqrt{-2})$ simultaneously governs both the endoscopic structure (through $\left(\frac{-2}{p}\right)$) and the dominant CM structure (through E_0). Equally mysterious is the endoscopic–motivic duality: two pure weight-0 motives combine to produce weight-(-1) CM content, paralleling how endoscopic contributions from smaller groups combine to produce the full automorphic spectrum.

The most promising paths forward are: (i) prove the motivic factorization from the spectral theory of P, which would simultaneously establish the weight dichotomy and sign formula; (ii) extend computation to identify the full family of abelian varieties that appear; and (iii) find the geometric object whose cohomology realizes n_p, which the motivic factorization now constrains to a precise mixed-motive structure involving Artin motives and h^1 of CM abelian varieties over $\mathbb{F}_2$.

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