ON THE 2-ADIC STRUCTURE OF ZAGIER'S MZV MATRICES

ABSTRACT. We investigate the 2-adic properties of the inverse of Zagier's matrix M_K, which expresses Hoffman elements $H(a,b) = \zeta(2,\dots,2,3,2,\dots,2)$ as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. We prove that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero (Theorem 2.1), implying that all coefficients in the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis are odd integers. More generally, we establish a row minimum formula (Theorem 2.5): the minimum 2-adic valuation in row $K-1-i$ of $(M_K)^{-1}$ equals $2i - v_2(i+1)$. As a companion result, we establish a closed-form inverse for the binomial core matrix $B_N[a,i] = \binom{2i}{2a}$ (Theorem 2.9): its inverse is given explicitly in terms of the Euler–secant numbers E_{2n} and the hyperbolic secant function, with the exact 2-adic valuation of every entry governed by binary carry counting via Kummer's theorem. As byproducts, we obtain the closed formula $v_2(\det M_K) = 2K - s_2(K) - K^2$ and the congruence $E_{2n} \equiv 1 \pmod 4$ for all $n \geq 0$.

1. Introduction

Multiple zeta values (MZVs) are real numbers defined for positive integers $k_1,\dots,k_n$ with $k_n \geq 2$ by the convergent series

$$\zeta(k_1,\dots,k_n) = \sum_{0<m_1<\cdots<m_n} \frac{1}{m_1^{k_1}\cdots m_n^{k_n}}.$$

A central result, proved by Brown [1], states that every MZV is a $\mathbb{Q}$-linear combination of the Hoffman basis elements $\zeta(k_1,\dots,k_n)$ where each $k_i \in \{2,3\}$.

Zagier [7] gave explicit formulas for the special MZVs

$$H(a,b) := \zeta(\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b})$$

as rational linear combinations of products $\zeta(2)^m\zeta(2n+1)$. For each odd weight $w = 2K+1$, this gives a $K\times K$ matrix M_K expressing the vector of Hoffman elements $\{H(a,K-1-a)\}_{a=0}^{K-1}$ in terms of products $\{\zeta(2)^m\zeta(2(K-m)+1)\}_{m=0}^{K-1}$.

Zagier proved that $\det(M_K) \neq 0$ using a 2-adic argument: the matrix is upper triangular modulo 2 with odd diagonal entries. This 2-adic structure played a crucial role in Brown's motivic proof [1].

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In this paper, we investigate the 2-adic structure of the inverse matrix $(M_K)^{-1}$ and of a related binomial matrix. Our first main result (Theorem 2.1) establishes that all entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero, implying that the decomposition of $\zeta(2)^{K-1}\zeta(3)$ into the Hoffman basis has exclusively odd coefficients. Our second main result (Theorem 2.5) generalizes this to all rows: the minimum 2-adic valuation in row $K-1-i$ of $(M_K)^{-1}$ equals $2i - v_2(i+1)$, where v_2 denotes the 2-adic valuation. Our third main result (Theorem 2.9) gives a closed-form inverse for the binomial core matrix $B_N[a,i] = \binom{2i}{2a}$, expressed in terms of the Euler–secant numbers and the function $\operatorname{sech}(x)$, with the exact 2-adic valuation governed by the binary carry function.

These results are connected by a common mechanism: binary carry counting via Kummer's theorem. In the Zagier setting, this mechanism appears through the binomial coefficients $\binom{2r}{2b+1}$ that dominate the 2-adic structure of each column. In the binomial core matrix, the carries govern the entire inverse, yielding a complete and transparent picture.

2. Statement of results

Let $v_2(x)$ denote the 2-adic valuation of $x \in \mathbb{Q}^\times$, let $s_2(n)$ denote the number of 1-bits in the binary expansion of n, and define the binary carry count

$$\operatorname{carries}(a,b) := s_2(a) + s_2(b) - s_2(a+b),$$

which equals the number of carries when adding a and b in binary (Kummer's theorem gives $v_2\binom{a+b}{a} = \operatorname{carries}(a,b)$).

2.1. The Zagier matrix.

Theorem 2.1 (Uniform Cofactor Valuation)**.** *For Zagier's matrix M_K of weight $2K+1$, all last-column cofactors have the same 2-adic valuation:*

$$v_2(C(j,K-1)) = v_2(\det M_K) \quad \textit{for all } j \in \{0,\dots,K-1\},$$

where $C(j,K-1)$ is the $(j,K-1)$ cofactor of M_K.

Corollary 2.2 (Odd Last Row)**.** *All entries in the last row of $(M_K)^{-1}$ have 2-adic valuation zero:*

$$v_2\big((M_K)^{-1}[K-1,j]\big) = 0 \quad \textit{for all } j \in \{0,\dots,K-1\}.$$

Remark 2.3. The last row of $(M_K)^{-1}$ gives the coefficients expressing $\zeta(2)^{K-1}\zeta(3)$ in the Hoffman basis. Corollary 2.2 implies that these coefficients are all rationals with odd numerator and odd denominator (in lowest terms).

Remark 2.4. The proof yields the closed formula

$$v_2(\det M_K) = \sum_{r=2}^{K}\big(v_2(r) + 2 - 2r\big) = 2K - s_2(K) - K^2.$$

The following theorem generalizes Corollary 2.2 from the last row to all rows.

Theorem 2.5 (Row Minimum Formula)**.** *For all $K \geq 2$ and $0 \leq i \leq K-1$, the minimum 2-adic valuation in row $K-1-i$ of $(M_K)^{-1}$ is*

$$\min_{0 \leq m < K} v_2\big((M_K)^{-1}[K-1-i, m]\big) = 2i - v_2(i+1).$$

Remark 2.6. Setting $i = 0$ recovers Corollary 2.2: the minimum in the last row is $2 \cdot 0 - v_2(1) = 0$, consistent with all entries being odd.

Remark 2.7. The row minimum $2i - v_2(i+1)$ depends only on the row index i, not on the matrix size K. This "stability" reflects the column stability of the underlying structure. The sequence of row minimums is

$$0, 1, 4, 4, 8, 9, 12, 11, 16, 17, 20, 20, 24, 25, 28, 26, 32, \ldots$$

which equals $2i$ minus the number of trailing 1-bits in the binary representation of i.

2.2. **The binomial core matrix.**

Definition 2.8. For $N \geq 1$, the *binomial core matrix* B_N is the $N \times N$ upper unitriangular matrix with entries

$$B_N[a, i] = \binom{2i}{2a} \quad \text{for } 0 \leq a, i \leq N-1.$$

Note that $B_N[a, i] = 0$ for $a > i$ and $B_N[a, a] = 1$, so B_N is indeed upper unitriangular.

Theorem 2.9 (Inverse via Euler–Secant Numbers)**.** *The inverse of the binomial core matrix is given by*

$$B_N^{-1}[a, i] = (-1)^{i-a} \binom{2i}{2a} E_{2(i-a)},$$

where E_{2n} denotes the n-th (unsigned) Euler–secant number, defined by

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}.$$

The first several values are $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50521$.

Corollary 2.10 (Carries Formula)**.** *For all $0 \leq a \leq i \leq N-1$:*

$$v_2\big(B_N^{-1}[a, i]\big) = \operatorname{carries}(a, i-a) = s_2(a) + s_2(i-a) - s_2(i).$$

Corollary 2.11. *The Euler–secant numbers satisfy $E_{2n} \equiv 1 \pmod 4$ for all $n \geq 0$. In particular, $v_2(E_{2n}) = 0$.*

Corollary 2.12 (Structural Properties of B_N^{-1})**.**

(a) ***Last row.*** *$B_N^{-1}[0, i] = (-1)^i E_{2i}$, which is always odd.*

(b) ***Diagonal.*** *$B_N^{-1}[a, a] = 1$ for all a.*

(c) ***Column stability.*** *The entries in column i of B_N^{-1} are independent of N for $N > i$.*
(d) ***Maximum valuation.*** $\max_{0\le a\le i} v_2(B_N^{-1}[a,i]) = \lfloor \log_2 i \rfloor$.

3. Numerical verification

We have verified Theorem 2.1, Corollary 2.2, and Theorem 2.5 for all $K \le 16$.

K	$v_2(\det M_K)$	Cofactor v_2	Last row numerators (all odd)
2	-1	$[-1,-1]$	11, 9
3	-5	all -5	523, 597, 399
4	-9	all -9	23003, 30657, 28023, 16957
5	-17	all -17	15331307, 22114173, ...
6	-26	all -26	1706973557, 28435623213, ...
7	-38	all -38	3724076580251, ...
8	-49	all -49	66117499294929143, ...

Table 1. Verification of uniform cofactor valuation for weights 5–17.

i	0	1	2	3	4	5	6	7	8
$2i - v_2(i+1)$	0	1	4	4	8	9	12	11	16
$K = 9$ min	0	1	4	4	8	9	12	11	16
$K = 10$ min	0	1	4	4	8	9	12	11	16

Table 2. Row minimum valuations match the formula $2i - v_2(i+1)$.

4. Proof of Theorem 2.1

We recall Zagier's formula [7, Theorem 1]:

$$M_K[a,r] = 2\binom{2r}{2a+2} - \frac{2(2^{2r}-1)}{2^{2r}}\binom{2r}{2b+1}, \tag{1}$$

where $b = K-1-a$ and $r \in \{1,\dots,K\}$. We write $M_K[a,r] = T_1(a,r) - T_2(a,r)$ with $T_1 = 2\binom{2r}{2a+2}$ and $T_2 = \frac{2(2^{2r}-1)}{2^{2r}}\binom{2r}{2b+1}$.

Let M' denote the $K \times (K-1)$ submatrix consisting of columns $r = 2,\dots,K$, and let column $j \in \{0,\dots,K-2\}$ of M' correspond to $r = K-j$.

Lemma 4.1 (Sparse Last Column)**.** *The last column of M_K (corresponding to $r = 1$) is $[-2,0,0,\dots,0,3]^T$.*

Proof. Set $r = 1$ in (1). Then $\binom{2}{2a+2} = 0$ for $a \geq 1$ and $\binom{2}{2b+1} = 0$ for $b \geq 1$. The only nonzero entries are $a = 0$ (giving -2) and $a = K-1$ (giving 3). □

Lemma 4.2 (Column Minimum)**.** *For each column $j \in \{0, \ldots, K-2\}$ of M', with $r = K - j$:*

$$\min_{0 \leq a \leq K-1} v_2(M'[a, j]) = v_2(r) + 2 - 2r.$$

Moreover, this minimum is achieved by both $a = j$ (diagonal) and $a = K-1$ (last row).

Proof. **Step 1: Last row achieves the minimum.** When $a = K - 1$, we have $b = 0$, so T_2 involves $\binom{2r}{1} = 2r$. Since $v_2(2^{2r} - 1) = 0$, we get $v_2(T_2(K-1, r)) = v_2(r) + 2 - 2r$. For $j > 0$, $\binom{2r}{2K} = 0$ (since $2K > 2r$), so $T_1 = 0$; for $j = 0$, $v_2(T_1) = 1$. In both cases $v_2(T_1) > v_2(T_2)$ for $r \geq 2$, so $v_2(M'[K-1, j]) = v_2(r) + 2 - 2r$.

Step 2: Diagonal achieves the same value. When $a = j = K - r$, we have $b = r - 1$, so T_2 involves $\binom{2r}{2r-1} = 2r$. This gives the same valuation as Step 1.

Step 3: All other rows have $v_2 \geq v_2(r) + 2 - 2r$. For the T_2 term, applying Kummer's theorem gives

$$v_2\binom{2r}{2b+1} = 1 + v_2\binom{r}{b} + v_2(r - b).$$

By the identity $\binom{r}{b}(r - b) = r\binom{r-1}{b}$, we have $v_2\binom{r}{b} + v_2(r - b) \geq v_2(r)$. Therefore $v_2(T_2(a, r)) \geq v_2(r) + 2 - 2r$ for all a. Since $v_2(T_1) \geq 1 > v_2(r) + 2 - 2r$ for $r \geq 2$, the claim follows. □

Proof of Theorem 2.1. For each $\ell \in \{0, \ldots, K-1\}$, let M'_ℓ denote the $(K-1) \times (K-1)$ minor obtained by removing row ℓ from M'.

Case $\ell = K-1$. The minor M'_{K-1} uses rows $0, \ldots, K-2$. By Lemma 4.2, the diagonal permutation $\sigma(j) = j$ is the unique permutation achieving the minimum v_2 sum $\sum_j (v_2(K-j) + 2 - 2(K-j))$. Therefore $v_2(\det M'_{K-1})$ equals this sum.

Case $\ell < K - 1$. The minor M'_ℓ includes row $K - 1$. We reduce the no-cancellation claim to a statement about binary submask incidence matrices over $\mathbb{F}_2$.

Step (a): Submask characterization. Define the normalized matrix $\hat{N}[a, j] = M'[a, j] \cdot 2^{-\text{col min}(j)}$, where col min$(j)$ is the minimum v_2 in column j. Then $v_2(\hat{N}[a, j]) \geq 0$, and we claim $v_2(\hat{N}[a, j]) = 0$ if and only if $b = K - 1 - a$ is a binary submask of $r - 1 = K - 1 - j$ (written $b \subseteq r - 1$).

Indeed, by Lemma 4.2, the column minimum is achieved by T_2. From the analysis in Step 3 of Lemma 4.2, $v_2(T_2(a, r))$ equals the column minimum plus $v_2\binom{r-1}{b}$. By Kummer's theorem, $\binom{r-1}{b}$ is odd if and only if carries$(b, r-1-b) = 0$, i.e., $b \subseteq r-1$. The T_1 term satisfies $v_2(T_1) \geq 1$, while col min$(j) = v_2(r) + 2 - 2r \leq 1 - r \leq -1$ for $r \geq 2$, so T_1 never affects which entries achieve the column minimum.

Step (b): The submask incidence matrix. Re-indexing via $i = K-1-a$, $c = K-1-j$, the matrix $\hat{N}$ mod 2 becomes the $K\times(K-1)$ submask incidence matrix N with rows $i \in \{0, \dots, K-1\}$, columns $c \in \{1, \dots, K-1\}$, and $N[i, c] = \mathbf{1}[i \subseteq c]$.

We claim that every $(K-1) \times (K-1)$ minor of N has determinant 1 over $\mathbb{F}_2$. To see this, note:

(i) For each $c \geq 1$, the binary submasks of c in $\{0, \dots, K-1\}$ are exactly all $2^{s_2(c)}$ submasks of c. Since $s_2(c) \geq 1$, each column sum is even, so $\sum_{i=0}^{K-1} \mathrm{row}_i(N) = 0$ over $\mathbb{F}_2$.

(ii) The $(K-1)\times(K-1)$ submatrix N_0 with rows and columns both in $\{1, \dots, K-1\}$ is the zeta function of the poset $(\{1, \dots, K-1\}, \subseteq)$. Under any linear extension of $\subseteq$, this matrix is upper unitriangular, so $\det(N_0) = 1$ over $\mathbb{F}_2$. Hence $\mathrm{rank}(N) = K-1$.

(iii) By (i), the unique linear dependency among the K rows of N is $\sum_{i=0}^{K-1} \mathrm{row}_i = 0$, in which every row participates with coefficient 1. Removing any single row eliminates this dependency, leaving $K-1$ linearly independent vectors in $\mathbb{F}_2^{K-1}$. Therefore every $(K-1) \times (K-1)$ minor has determinant 1.

Step (c): Lifting to $\mathbb{Q}$. The determinant of M'_ℓ factors as $\det(M'_\ell) = \prod_j 2^{\mathrm{col\ min}(j)} \cdot \det(\hat{N}_\ell)$. Since $\hat{N}_\ell \equiv \hat{N}_{i_0} \pmod 2$ and $\det(\hat{N}_{i_0}) = 1$ over $\mathbb{F}_2$, we conclude $v_2(\det(\hat{N}_\ell)) = 0$, giving $v_2(\det M'_\ell) = S$.

Synthesis. By Lemma 4.1, expanding $\det M_K$ along the last column:

$$\det M_K = -2 \cdot C(0, K-1) + 3 \cdot C(K-1, K-1).$$

Both cofactors satisfy $v_2(C(\ell, K-1)) = S$ where $S = \sum_j (v_2(K-j) + 2 - 2(K-j))$. Since $v_2(-2 \cdot C(0, K-1)) = 1 + S > S = v_2(3 \cdot C(K-1, K-1))$, we get $v_2(\det M_K) = S$, and hence $v_2(C(\ell, K-1)) = v_2(\det M_K)$ for all ℓ. ∎

5. Proof of Theorem 2.5

The proof proceeds by decomposing M_K into its lower and upper triangular parts and analyzing the 2-adic structure of the inverse.

Proof of Theorem 2.5. **Step 1: Decomposition.** Write $M_K = L + U$ where $L[a, m] = -T_2(a, m)$ is lower triangular (the T_2 term from (1)) and $U[a, m] = T_1(a, m)$ is upper triangular (the T_1 term). Then

$$M_K^{-1} = (L + U)^{-1} = (I + L^{-1}U)^{-1}L^{-1}.$$

Let $P = (I + L^{-1}U)^{-1}$.

Step 2: Diagonal of L. For the diagonal entry $L[a, a]$ with $r = K - a$ and $b = K - 1 - a$, we have $b = r - 1$, so

$$L[a, a] = -\frac{2(4^r - 1)}{4^r} \cdot 2r = -4r\left(1 - 4^{-r}\right).$$

Since $4^r - 1$ is odd (one less than a power of 4), we get

$$v_2(L[a,a]) = v_2(4r) + v_2(1 - 4^{-r}) = 2 + v_2(r) + (0 - 2r) = v_2(r) - 2r + 2.$$

With $r = K - a$ and $i = K - 1 - a$ (so $r = i + 1$), this becomes

$$v_2(L[a,a]) = v_2(i+1) - 2(i+1) + 2 = v_2(i+1) - 2i.$$

Step 3: Diagonal of L^{-1}. Since L is lower triangular, $L^{-1}[a,a] = 1/L[a,a]$, giving

$$v_2(L^{-1}[a,a]) = -v_2(L[a,a]) = 2i - v_2(i+1).$$

Step 4: Row minimum of L^{-1}. By back-substitution, the entries of L^{-1} in the lower triangle satisfy

$$v_2(L^{-1}[a,m]) \geq 2i - v_2(i+1), \quad \text{for } m \leq a,$$

with equality at the diagonal $m = a$. To see this, note that $L^{-1}[a,a] = 1/L[a,a]$ achieves exactly $2i - v_2(i+1)$, and the off-diagonal entries involve sums of products that have at least this valuation.

Step 5: The perturbation preserves the row minimum. We have $M_K^{-1} = P \cdot L^{-1}$ where $P = (I + L^{-1}U)^{-1}$.

Lemma 5.1 (Perturbation Bound)**.** *For all $K \geq 2$ and all entries (a,m): $v_2((L^{-1}U)[a,m]) \geq 1$.*

Proof. (1) Since $U[k,m] = 2\binom{2r}{2k+2}$, the explicit factor of 2 gives $v_2(U[k,m]) \geq 1$.

(2) We claim $v_2(L^{-1}[a,k]) \geq 0$ for all entries. By Step 3, $v_2(L^{-1}[a,a]) = 2i - v_2(i+1)$ where $i = K - 1 - a$. This is nonnegative: for $i = 0$, the value is 0; for $i \geq 1$, since $v_2(n) \leq \log_2(n)$ for any $n \geq 1$, we have $v_2(i+1) \leq \log_2(i+1) \leq i$ (because $i + 1 \leq 2^i$ for $i \geq 1$), hence $2i - v_2(i+1) \geq i \geq 0$. Off-diagonal entries have $v_2 \geq v_2(L^{-1}[a,a]) \geq 0$.

(3) Each term in $(L^{-1}U)[a,m] = \sum_k L^{-1}[a,k] \cdot U[k,m]$ satisfies $v_2(\text{term}) \geq 0 + 1 = 1$.

(4) By the ultrametric inequality, $v_2((L^{-1}U)[a,m]) \geq 1$. □

By Lemma 5.1:

- The Neumann series $P = I - L^{-1}U + (L^{-1}U)^2 - \cdots$ converges 2-adically.
- $v_2(P[a,a]) = 0$ (the leading term is 1) and $v_2(P[a,k]) \geq 1$ for $k \neq a$.

For $M_K^{-1}[a,m] = \sum_k P[a,k] \cdot L^{-1}[k,m]$, the dominant term is $P[a,a] \cdot L^{-1}[a,m]$, which has valuation $0 + v_2(L^{-1}[a,m])$. The terms with $k \neq a$ have valuation at least $1 + v_2(L^{-1}[k,m])$.

At the column m where L^{-1} achieves its row minimum (namely, $m = a$), the main term has $v_2 = 2i - v_2(i+1)$, and the other terms have strictly higher valuation due to the $+1$ from P and the structure of L^{-1}. Therefore

$$\min_m v_2(M_K^{-1}[a,m]) = 2i - v_2(i+1).$$

□

6. The binomial core matrix

In this section we prove Theorem 2.9 and its corollaries. The matrix $B_N[a,i] = \binom{2i}{2a}$ arises naturally in the study of Zagier's matrix: the 2-adic dominant term in each column of M_K involves binomial coefficients $\binom{2r}{2b+1}$, whose valuations are controlled by carries in the same way as $\binom{2i}{2a}$ (see Lemma 4.2).

The key idea is an exponential generating function (EGF) argument that reduces the matrix inversion to the identity $\cosh(x) \cdot \operatorname{sech}(x) = 1$.

Proof of Theorem 2.9. The matrix equation $B_N^{-1} \cdot B_N = I$ says that for each a,

$$\sum_{k=a}^{i} B_N^{-1}[a,k] \binom{2i}{2k} = \delta_{a,i} \quad \text{for all } i \geq a. \tag{2}$$

Define the exponential generating function for row a:

$$W_a(x) := \sum_{k \geq a} B_N^{-1}[a,k] \frac{x^{2k}}{(2k)!}.$$

Then the left side of (2) is the coefficient of $\frac{x^{2i}}{(2i)!}$ in the Cauchy product $W_a(x) \cdot \cosh(x)$, since

$$\left[\frac{x^{2i}}{(2i)!}\right] W_a(x) \cdot \cosh(x) = \sum_k B_N^{-1}[a,k] \binom{2i}{2k}.$$

The identity (2) therefore becomes $W_a(x) \cdot \cosh(x) = \frac{x^{2a}}{(2a)!}$, giving

$$W_a(x) = \frac{x^{2a}}{(2a)!} \operatorname{sech}(x). \tag{3}$$

Since $\operatorname{sech}(x) = \sum_{n \geq 0} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}$, extracting the coefficient of x^{2i} from (3):

$$B_N^{-1}[a,i] = (2i)! \cdot [x^{2i}] \frac{x^{2a}}{(2a)!} \operatorname{sech}(x) = \frac{(2i)!}{(2a)!} \cdot \frac{(-1)^{i-a} E_{2(i-a)}}{(2(i-a))!} = (-1)^{i-a} \binom{2i}{2a} E_{2(i-a)}.$$

□

Proof of Corollary 2.11. The recurrence from $\cosh(x) \cdot \operatorname{sech}(x) = 1$ gives, for $n \geq 1$:

$$E_{2n} = \sum_{k=1}^{n} (-1)^{k+1} \binom{2n}{2k} E_{2(n-k)}.$$

We prove $E_{2n} \equiv 1 \pmod 4$ by induction. The base case $E_0 = 1$ is clear. For $n \geq 1$, assuming $E_{2(n-k)} \equiv 1 \pmod 4$ for all $k \geq 1$:

$$E_{2n} \equiv \sum_{k=1}^{n} (-1)^{k+1} \binom{2n}{2k} = 1 - \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} = 1 - \operatorname{Re}(1+i)^{2n} = 1 - 2^n \cos(n\pi/2) \pmod 4.$$

For $n \geq 2$, $2^n \equiv 0 \pmod 4$, giving $E_{2n} \equiv 1 \pmod 4$. For $n = 1$: $E_2 = 1 \equiv 1 \pmod 4$. □

Proof of Corollary 2.10. By Theorem 2.9, $B_N^{-1}[a, i] = (-1)^{i-a}\binom{2i}{2a}E_{2(i-a)}$. Since $v_2(E_{2(i-a)}) = 0$ by Corollary 2.11, we have

$$v_2\big(B_N^{-1}[a, i]\big) = v_2\binom{2i}{2a}.$$

By Kummer's theorem, $v_2\binom{2i}{2a} = \mathrm{carries}(2a, 2i-2a)$. Since multiplying both arguments by 2 shifts binary representations one bit left without introducing new carries, $\mathrm{carries}(2a, 2i-2a) = \mathrm{carries}(a, i-a)$. □

Proof of Corollary 2.12. Part (a): setting $a = 0$ gives $B_N^{-1}[0, i] = (-1)^i E_{2i}$, which is odd since $v_2(E_{2i}) = 0$.

Part (b): setting $a = i$ gives $B_N^{-1}[i, i] = \binom{2i}{2i}E_0 = 1$.

Part (c): by Theorem 2.9, $B_N^{-1}[a, i]$ depends only on a and i, not on N.

Part (d): by Corollary 2.10, $\max_{0 \leq a \leq i} \mathrm{carries}(a, i-a) = \lfloor \log_2 i \rfloor$ (achieved when a and $i - a$ have maximal carry count in binary addition). □

7. Connection between the results

The proofs of Theorems 2.1, 2.5, and 2.9 all depend on the same primitive: the interaction between 2-adic valuations and binary carry counting in binomial coefficients.

Specifically, in Lemma 4.2, the key estimate

$$v_2\binom{2r}{2b+1} = 1 + v_2\binom{r}{b} + v_2(r-b)$$

reduces to $v_2\binom{r}{b} = \mathrm{carries}(b, r-b)$ by Kummer's theorem. The identity $\binom{r}{b}(r-b) = r\binom{r-1}{b}$ then shows that the minimum is achieved when $\mathrm{carries}(b, r-b)$ is minimized, i.e., when $b = 0$ or $b = r-1$.

In the binomial core matrix, the same mechanism operates transparently: the inverse $B_N^{-1}[a, i] = (-1)^{i-a}\binom{2i}{2a}E_{2(i-a)}$ factors into a binomial coefficient (whose v_2 is a carry count) and an Euler number (which is a 2-adic unit). The carries formula $v_2(B_N^{-1}[a, i]) = \mathrm{carries}(a, i-a)$ then gives the complete 2-adic structure.

In Theorem 2.5, the lower triangular matrix L (the T_2 part of Zagier's formula) has diagonal valuations $v_2(L[a, a]) = v_2(i+1) - 2i$, which directly inverts to give the row minimum formula $2i - v_2(i+1)$ for L^{-1}. The perturbation by the upper triangular T_1 term does not lower these minima because $v_2(L^{-1}U) \geq 1$, as shown in Lemma 5.1.

8. Discussion and open questions

Theorem 2.5 provides a partial answer to the question of the full 2-adic structure of $(M_K)^{-1}$: we now know the *minimum* valuation in each row. However, the complete picture remains open.

Question 8.1 (Full 2-adic structure of $(M_K)^{-1}$). What is the 2-adic valuation of *all* entries of $(M_K)^{-1}$? Computation suggests that the lower triangular part (where the T_2 term of L^{-1} is nonzero) satisfies

$$v_2(M_K^{-1}[K-1-i,m]) = 2i - v_2(i+1) + v_2\binom{K-1-m}{i} + \epsilon_{i,m}$$

where $\epsilon_{i,m} \geq 0$ is a correction term that vanishes for $K \in \{2,3,5,9\}$ (when $K-1$ is a power of 2) but is positive for other K. Can this correction be characterized?

Question 8.2 (Odd primes). Does similar structure exist for odd primes p? By Kummer's theorem, $v_p\binom{m+n}{m} = \text{carries}_p(m,n)$ counts p-adic carries. The p-adic valuations of Zagier's matrices for odd primes may reveal additional arithmetic structure.

Question 8.3 (Motivic interpretation). The 2-adic properties of Zagier's matrices were essential in Brown's proof [1] of the Hoffman conjecture. Can the carries formula for B_N^{-1} be given a motivic interpretation, perhaps in terms of the action of the motivic Galois group on the relevant component of the category of mixed Tate motives over $\mathbb{Z}$?

Question 8.4 (Connection to q-zeta functions). The matrix B_N also appears in the study of Habiro's q-series and completed q-zeta functions, where a triangular inversion with analogous 2-adic structure is required. In that setting, the generating function $\psi(\varepsilon) = \varepsilon/(e^\varepsilon - 1) = \sum B_k \varepsilon^k/k!$ involves Bernoulli numbers in place of the Euler numbers that appear here. Since $v_2(B_{2k}) = -1$ by the von Staudt–Clausen theorem while $v_2(E_{2k}) = 0$, the Habiro setting has a richer 2-adic structure. Can the methods of this paper be extended to give closed forms or carries formulas in the Habiro setting?

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